FORMULA SHEETS

1. **Radial, velocity, and acceleration vectors in 2-D polar coordinates** (r, θ):

$$\begin{aligned}\vec{r} &= r\hat{e}_r;\\ \vec{v} &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta;\\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta.\end{aligned}$$

2. **Radial, velocity, and acceleration vectors in cylindrical coordinates** (R, ϕ, z):

$$\begin{aligned}\vec{r} &= R\hat{e}_R + z\hat{e}_z;\\ \vec{v} &= \dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z;\\ \vec{a} &= (\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z.\end{aligned}$$

3. **Radial, velocity, and acceleration vectors in spherical polar coordinates** (r, θ, ϕ):

$$\begin{aligned}\vec{r} &= r\hat{e}_r;\\ \vec{v} &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\hat{e}_\phi;\\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{e}_\theta\\ &\qquad + (2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{e}_\phi.\end{aligned}$$

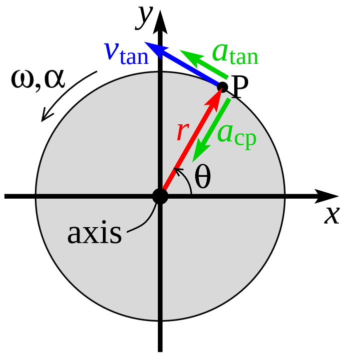
4. **Circular motion**: For a wheel rotating about a fixed axis at angular speed, $\dot{\theta} = \omega$, and angular acceleration, $\ddot{\theta} = \alpha$,

$$v_{\text{tan}} = r\omega \qquad \text{and} \qquad a_{\text{tan}} = r\alpha,$$

where: v_{tan}, a_{tan} are the tangential velocity, acceleration of a point P on the rim of the wheel relative to its axis; and r is the radius of the wheel.

In addition to a_{tan}, point P has a centripetal acceleration directed toward the axis given by:

$$a_{\text{cp}} = \frac{v_{\text{tan}}^2}{r} = \omega^2 r.$$

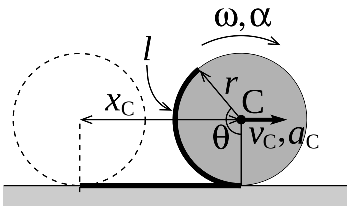


5. **Centripetal acceleration:** For an object moving at speed v in a circular path of radius r, its centripetal acceleration directed toward the centre of curvature is,

$$a_{\text{cp}} = \frac{v^2}{r} = \omega^2 r,$$

where $\omega = v/r$ is the angular speed.

6. **Rolling motion (no-slip condition)**: Let C correspond to the axle of the wheel. Then, for the wheel to roll without slipping,



$$x_{\mathrm{C}} = l = r\theta; \qquad v_{\mathrm{C}} = r\omega; \qquad a_{\mathrm{C}} = r\alpha,$$

where: x_{C}, v_{C}, and a_{C} are the distance traveled, translational speed, and acceleration of the axle respectively; r is the radius of the wheel; $\omega = \dot{\theta}$, $\alpha = \ddot{\theta}$ are the angular speed, acceleration of the wheel about C.

7. Variations of **Newton's Second Law** for rectilinear (1-D) motion:

$$\sum F = ma = m\ddot{x} = m\dot{v} = m\frac{dv}{dx}v = \frac{m}{2}\frac{dv^2}{dx}.$$

8. **Equations of kinematics** for constant linear acceleration, a, or angular acceleration, α:

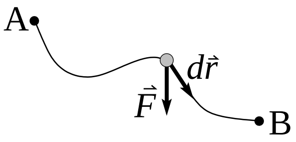
$$\begin{aligned} x &= x_0 + v_0 t + \tfrac{1}{2}at^2 & \qquad \theta &= \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \\ v &= v_0 + at & \omega &= \omega_0 + \alpha t \\ v^2 &= v_0^2 + 2a(x - x_0) & \omega^2 &= \omega_0^2 + 2\alpha(\theta - \theta_0) \end{aligned}$$

9. **Frictional forces** (Coulomb model):

$$f_{\mathrm{k}} = \mu_{\mathrm{k}} N; \qquad f_{\mathrm{s}} \leq \mu_{\mathrm{s}} N; \qquad D = c_1 v + c_2 v^2,$$

where f_{k}, f_{s}, and D are the kinetic friction force, static friction force, and "air drag" respectively, N is the normal force, v is the velocity of a particle through the air, and where the remaining quantities are coefficients.

10. **Work done by a force along a path**:



$$W = \int_{\mathrm{A}}^{\mathrm{B}} \vec{F} \cdot d\vec{r}.$$

If $\vec{F}$ is conservative, then W depends only on the end points A and B, independent of the path chosen between them.

11. **Work-kinetic theorem**: Work done by all external forces along a path between points A and B is equal to the difference in kinetic energy at points B and A:

$$\sum W = \Delta K \quad \Rightarrow \quad \sum_i \int_{\mathrm{A}}^{\mathrm{B}} \vec{F}_{\mathrm{ext},i} \cdot d\vec{r} = K_{\mathrm{B}} - K_{\mathrm{A}}.$$

12. **Conservative forces:** If $\vec{F} = \vec{F}(\vec{r})$ and $\nabla \times \vec{F} = 0$, $\vec{F}$ is *conservative* and there exists a scalar function, $U(\vec{r})$, the *potential energy*, such that,

$$\vec{F} \;=\; -\nabla U(\vec{r}).$$

For rectilinear (1-D) motion, if F is a function of x only,

$$F(x) \;=\; -\frac{dU}{dx} \qquad \text{and} \qquad U(x) - U(x_0) \;=\; -\int_{x_0}^{x} F(x')dx'.$$

13. **Conservation of mechanical energy**: In a system where all forces are conservative, mechanical energy is conserved:

$$E \;=\; K + U \;=\; \tfrac{1}{2}mv^2 + U(\vec{r}) \;=\; \text{constant},$$

where K and U are respectively the kinetic and potential energies of the system.

For rectilinear (1-D) motion, if F is a function of x only, mechanical energy is conserved:

$$E \;=\; K + U \;=\; \tfrac{1}{2}m\dot{x}^2 + U(x) \;=\; \text{constant},$$

where K and U are respectively the kinetic and potential energies of the system.

14. **Simple harmonic oscillator (SHO):** Any system of mass m whose differential equation of motion has the form,

$$\frac{d^2x}{dt^2} + \omega_0^2 x \;=\; 0,$$

or whose total energy has the form,

$$E \;=\; K + U \;=\; \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \;=\; \text{constant},$$

describes an undamped, undriven SHO oscillating at angular frequency ω_0 ($\mathrm{rad\,s^{-1}}$).

The solution to either of these equations can be written in three forms:

$$x(t) \;=\; A\cos\omega_0 t + B\sin\omega_0 t \;=\; x_0\cos(\omega_0 t - \phi_0) \;=\; ae^{i\omega_0 t} + be^{-i\omega_0 t},$$

where: $A, B \in \mathbb{R}$ are constants of integration set by initial conditions;
x_0 = amplitude of oscillation;
ϕ_0 = phase of oscillation;
$a, b \in \mathbb{C}$ are constants of integration set by initial conditions.

15. **Hooke's Law spring**: an example of a simple harmonic oscillator.

When stretched/compressed a distance x, a *Hooke's spring* exerts a *restoring force*, $F = -kx$, where k is a constant and the minus sign means the force opposes the distortion.

Coupled with Newton's second law,

$$F = m\frac{d^2x}{dt^2} = -kx \quad \Rightarrow \quad \ddot{x} + \frac{k}{m}x = 0,$$

which describes a simple harmonic oscillator whose equation of motion is,

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t,$$

where $\omega_0 = \sqrt{k/m}$ is the frequency of oscillation.

Potential energy stored in a Hooke's spring,

$$U(x) - U_0 = -\int_0^x F(x')dx' = k\int_0^x x'dx' \quad \Rightarrow \quad U(x) = \frac{1}{2}kx^2,$$

taking $U_0 = U(0) = 0$. Thus, total energy of a Hooke's oscillator in motion is:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

16. **Period of oscillation:** If the angular frequency of oscillation is ω ($\mathrm{rad\,s^{-1}}$), the period of oscillation in seconds is:

$$T = \frac{2\pi}{\omega}.$$

17. **Damped harmonic oscillator:** The differential equation of motion for a damped harmonic oscillator is,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \tag{1}$$

where: γ = damping coefficient;
ω_0 = angular frequency of the undamped oscillator.

For initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$, the solution to equation (1) is:

$$x(t) = \begin{cases} x_0 e^{-\gamma t}\left(\cosh qt + \dfrac{\gamma}{q}\sinh qt\right), & \gamma > \omega_0 \text{ (overdamped);} \\ x_0 e^{-\gamma t}(1 + \gamma t), & \gamma = \omega_0 \text{ (critically damped);} \\ x_0 e^{-\gamma t}\,\dfrac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \theta_0), & \gamma < \omega_0 \text{ (underdamped);} \\ x_0\cos\omega_0 t, & \gamma = 0 \text{ (undamped),} \end{cases}$$

where: x_0 = initial displacement of oscillator;
$q = \sqrt{\gamma^2 - \omega_0^2}$;
$\omega_\mathrm{d} = \sqrt{\omega_0^2 - \gamma^2}$ = angular frequency of damped oscillator;
$\theta_0 = \sin^{-1}\dfrac{\gamma}{\omega_0}$ = phase lag of damped oscillator.

18. **Quality factor of a damped harmonic oscillator**, Q_{d}, is a measure of how slowly the total energy of an oscillator is lost to damping. For underdamped systems,

$$Q_{\mathrm{d}} \approx \frac{\omega_{\mathrm{d}}}{2\gamma},$$

where: ω_{d} = angular frequency of the damped oscillator;

$\gamma = \sqrt{\omega_0^2 - \omega_{\mathrm{d}}^2}$, the damping coefficient;

ω_0 = angular frequency of the undamped oscillator.

19. **Driven, damped harmonic oscillator:** The differential equation of motion for a damped harmonic oscillator driven by a sinusoidally-varying force is,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t, \tag{1}$$

where: γ = damping coefficient;

ω_0 = natural angular frequency of the undamped oscillator;

m = mass of the oscillator;

F_0 = amplitude of the driving force;

ω = angular frequency of the driving force.

For initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, the solution to equation (1) for an underdamped ($\gamma < \omega_0$) oscillator is:

$$x(t) = \underbrace{-Ae^{-\gamma t}\,\frac{\omega_0}{\omega_{\mathrm{d}}}\cos(\omega_{\mathrm{d}}t - \theta)}_{x_{\mathrm{h}}(t)} + \underbrace{A\cos(\omega t - \phi)}_{x_{\mathrm{p}}(t)},$$

where: $x_{\mathrm{h}}(t)$ = transient term (homogeneous solution) which dies away with time;

$x_{\mathrm{p}}(t)$ = steady-state term (particular solution) remaining after transient disappears,

and where: $A = \dfrac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$ = amplitude of steady-state oscillation;

$\omega_{\mathrm{d}} = \sqrt{\omega_0^2 - \gamma^2}$ = natural angular frequency of damped oscillator;

$\tan\theta = \dfrac{\gamma}{\omega_{\mathrm{d}}}\,\dfrac{\omega_0^2 + \omega^2}{\omega_0^2 - \omega^2}$ = phase lag of transient term;

$\tan\phi = \dfrac{2\gamma\omega}{\omega_0^2 - \omega^2}$ = phase lag of steady-state term.

20. **Driven, damped harmonic oscillator, steady state:** As $t \to \infty$, equation of motion for a damped oscillator driven by a force $F(t) = F_0\cos\omega t$ is:

$$x(t) = A\cos(\omega t - \phi),$$

$$\begin{aligned}
\text{where: } A &= \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} = \text{amplitude of steady-state oscillation;}\\
\tan\phi &= \frac{2\gamma\omega}{\omega_0^2-\omega^2} = \text{phase lag of steady-state oscillation.}\\
m &= \text{mass of the oscillator;}\\
\omega_0 &= \text{natural angular frequency of the undamped oscillator;}\\
\gamma &= \text{damping coefficient.}
\end{aligned}$$

21. **Resonance:** Maximum amplitude of a driven, damped oscillator is: $A_{\mathrm{max}} = \dfrac{F_0}{2m\gamma\omega_{\mathrm{d}}}$, when driven at the *resonant frequency*,

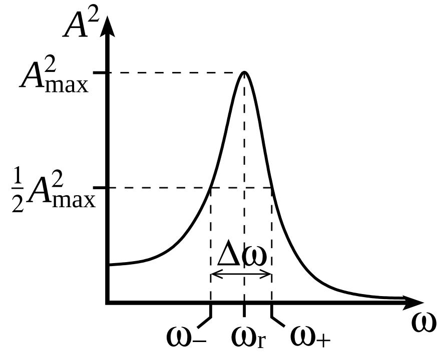
$$\omega_{\mathrm{r}} = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\omega_{\mathrm{d}}^2 - \gamma^2},$$

$$\begin{aligned}
\text{where: } F_0 &= \text{amplitude of the driving force;}\\
m &= \text{mass of the oscillator;}\\
\gamma &= \text{damping coefficient;}\\
\omega_{\mathrm{d}} &= \sqrt{\omega_0^2-\gamma^2} = \text{natural angular frequency of damped oscillator;}\\
\omega_0 &= \text{natural angular frequency of undamped oscillator.}
\end{aligned}$$

Sharpness of resonance is defined as,

$$S = \frac{\Delta\omega}{\omega_{\mathrm{r}}} = \frac{\omega_+ - \omega_-}{\omega_{\mathrm{r}}} \approx \frac{1}{Q_{\mathrm{d}}} \text{ for } \gamma \ll \omega_0,$$

$$\begin{aligned}
\text{where: } \Delta\omega &= \text{full-width-half-maximum (FWHM);}\\
\omega_\pm^2 &= \omega_{\mathrm{r}}^2 \pm 2\gamma\omega_{\mathrm{d}} \text{ (note the squares!)}\\
&= \text{driving frequencies where } A^2 = \tfrac{1}{2}A_{\mathrm{max}}^2;\\
Q_{\mathrm{d}} &= \frac{\omega_{\mathrm{d}}}{2\gamma} = \text{quality factor.}
\end{aligned}$$



22. **Projectile motion in an inertial frame of reference**. In the absence of air resistance, the position components of a projectile are given by:

$$\begin{aligned}
x(t) &= x_0 + (v_0\cos\alpha)t; && \text{(horizontal component)}\\
z(t) &= z_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2, && \text{(vertical component)}
\end{aligned}$$

$$\begin{aligned}
\text{where: } (x_0, 0, z_0) &= \text{initial position of the projectile;}\\
v_0 &= \text{initial speed of the projectile;}\\
\alpha &= \text{angle between } \vec{v}_0 \text{ and the horizontal in the } x\text{-}z \text{ plane.}
\end{aligned}$$

By setting $x_0 = 0$ and eliminating t between the x- and z-components, one gets the trajectory equation:

$$z(x) \;=\; z_0 + x\tan\alpha - \frac{g}{2v_0^2\cos^2\!\alpha}x^2.$$

Including air resistance of the form: $\vec{D} = -m\gamma\vec{v}$, where γ is a constant and m is the mass of the projectile, the position components and trajectory become:

$$x(t) \;=\; \frac{v_0\cos\alpha}{\gamma}\big(1 - e^{-\gamma t}\big); \qquad z(t) \;=\; \frac{1}{\gamma}\left(v_0\sin\alpha + \frac{g}{\gamma}\right)\big(1 - e^{-\gamma t}\big) - \frac{gt}{\gamma}.$$

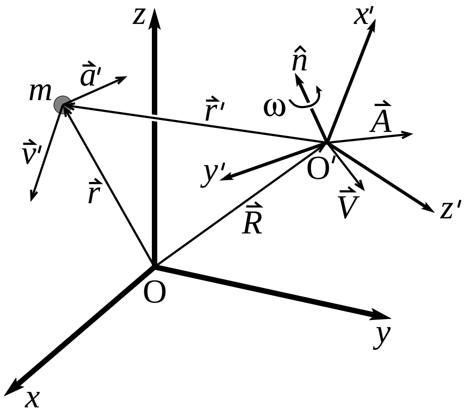
$$z(x) \;=\; \frac{x}{v_0\cos\alpha}\left(v_0\sin\alpha + \frac{g}{\gamma}\right) + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x}{v_0\sin\alpha}\right).$$

23. **Non-inertial frames of reference and Coriolis' theorem**: Kinematical quantities in an inertial (O) and non-inertial (O′) frames of reference are related by:

$$\vec{r} \;=\; \vec{r}\,' + \vec{R};$$

$$\vec{v} \;=\; \vec{v}\,' + \vec{\omega}\times\vec{r}\,' + \vec{V};$$

$$\vec{a} \;=\; \vec{a}\,' + \dot{\vec{\omega}}\times\vec{r}\,' + 2\,\vec{\omega}\times\vec{v}\,' + \vec{\omega}\times(\vec{\omega}\times\vec{r}\,') + \vec{A},$$



where: $\vec{r}$ = position of m relative to O;
$\vec{r}\,'$ = position of m relative to O′;
$\vec{R}$ = position of O′ relative to O;
$\vec{v}$ = velocity of m relative to O;
$\vec{v}\,'$ = velocity of m relative to O′;
$\vec{V}$ = velocity of O′ relative to O;
$\vec{\omega}$ = angular velocity of O′ relative to O about a fixed axis (*e.g.*, $\hat{n}$);
$\vec{a}$ = acceleration of m relative to O;
$\vec{a}\,'$ = acceleration of m relative to O′; and
$\vec{A}$ = acceleration of O′ relative to O,

and where the *inertial* accelerations are defined as follows:

$-\dot{\vec{\omega}}\times\vec{r}\,'$ = *transverse* acceleration; $\qquad -2\,\vec{\omega}\times\vec{v}\,'$ = *Coriolis* acceleration;
$-\vec{\omega}\times(\vec{\omega}\times\vec{r}\,')$ = *centrifugal* acceleration; $\qquad \vec{A}$ = *translational* acceleration.

Coriolis' theorem: $\boxed{\vec{F}' \;=\; m\vec{a}\,' \;=\; \vec{F} - m\,\dot{\vec{\omega}}\times\vec{r}\,' - 2m\,\vec{\omega}\times\vec{v}\,' - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}\,') - m\vec{A},}$

where:

$\vec{F}'$ = forces observed in O′; $\qquad \vec{F} = m\vec{a}$, real forces observed in O;

$$-m\,\dot{\vec\omega}\times\vec r' \;=\; \textit{transverse}\text{ force, }\vec F_\perp; \qquad -2m\,\vec\omega\times\vec v' \;=\; \textit{Coriolis}\text{ force, }\vec F_{\rm Cor};$$
$$-m\,\vec\omega\times(\vec\omega\times\vec r') \;=\; \textit{centrifugal}\text{ force, }\vec F_{\rm cent}; \qquad -m\vec A \;=\; \textit{translational}\text{ force, }\vec F_{\rm tran}.$$

24. **Projectile motion near the surface of the earth** (a rotating frame of reference): In the absence of air resistance, the position components of a projectile are given by:

$$\begin{aligned} x'(t) &= x'_0+\dot x'_0t-\omega t^2(\dot z'_0\cos\lambda-\dot y'_0\sin\lambda)+\tfrac13\omega g t^3\cos\lambda; && \text{(east-west)}\\ y'(t) &= y'_0+\dot y'_0t-\omega\dot x'_0t^2\sin\lambda; && \text{(north-south)}\\ z'(t) &= z'_0+\dot z'_0t-\tfrac12 gt^2+\omega\dot x'_0t^2\cos\lambda, && \text{(up-down)}\end{aligned}$$

where (x'_0, y'_0, z'_0) is the initial position of the projectile, $(\dot x'_0, \dot y'_0, \dot z'_0)$ is its initial velocity, ω is the rotation speed of the earth (7.292×10^{-5} rad s^{-1}), and λ is the latitude.

25. **MacLauren series** for some common functions:

$$\sin x \;=\; x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots; \qquad \cos x \;=\; 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots;$$
$$e^x \;=\; 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\dots; \qquad \ln(1+x) \;=\; x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots.$$

26. **Hyperbolic trigonometric functions and identities**:

$$\sinh(x) \;=\; \frac{e^x-e^{-x}}{2}; \qquad \cosh(x) \;=\; \frac{e^x+e^{-x}}{2};$$
$$\frac{d}{dx}\sinh(x) \;=\; \cosh(x); \qquad \frac{d}{dx}\cosh(x) \;=\; \sinh(x);$$
$$\cosh^2(x)-\sinh^2(x) \;=\; 1.$$

27. Some **trigonometric identities**:

$$\sin\theta \;=\; \sqrt{\frac{1-\cos2\theta}{2}}; \qquad \sin(\theta+\phi) \;=\; \sin\theta\cos\phi+\cos\theta\sin\phi;$$
$$\sin2\theta \;=\; 2\sin\theta\cos\theta;$$
$$\cos\theta \;=\; \sqrt{\frac{1+\cos2\theta}{2}}; \qquad \cos(\theta+\phi) \;=\; \cos\theta\cos\phi-\sin\theta\sin\phi;$$
$$\cos2\theta \;=\; \cos^2\theta-\sin^2\theta;$$
$$\tan\theta \;=\; \cot\Big(\frac{\pi}{2}-\theta\Big); \qquad \tan(\theta+\phi) \;=\; \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi};$$
$$\cot\theta \;=\; \tan\Big(\frac{\pi}{2}-\theta\Big); \qquad \cot(\theta+\phi) \;=\; \frac{\cot\theta\cot\phi-1}{\cot\theta+\cot\phi}.$$

28. **Small angle approximation:** For $\theta \ll 1$ where θ is in radians,

$$\sin\theta \approx \tan\theta \approx \theta \qquad \text{and} \qquad \cos\theta \approx 1.$$

29. **Integration by parts**:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

30. **Solving a first-order ODE by separation of variables**: A first order ordinary differential equation of the form,

$$\frac{dy}{dx} = \frac{f(x)}{g(y)},$$

where $f(x)$ and $g(y)$ are arbitrary functions is *separable*, and can be rewritten as,

$$g(y)\,dy = f(x)\,dx \quad \Rightarrow \quad \int g(y)\,dy = \int f(x)\,dx.$$

31. Solving an **inhomogeneous differential equation**: For a differential equation of the form:

$$a\frac{d^2y(x)}{dx^2} + b\frac{dy(x)}{dx} + cy(x) = d, \tag{1}$$

where a, b, c, and d are constants, first solve the *homogeneous* equation:

$$a\frac{d^2y_{\rm h}}{dx^2} + b\frac{dy_{\rm h}}{dx} + cy_{\rm h} = 0, \tag{2}$$

to find two linearly independent solutions, $y_1(x)$ and $y_2(x)$. The general solution to equation (2) is then,

$$y_{\rm h}(x) = Ay_1(x) + By_2(x),$$

where A and B are constants of integration.

Then find a *particular* solution, $y_{\rm p}$, which is *anything* that solves equation (1); $y_{\rm p} = d/c$ will do. The general solution to equation (1) is then:

$$y(x) = y_{\rm h} + y_{\rm p} = Ay_1(x) + By_2(x) + \frac{d}{c},$$

to which boundary/initial conditions are applied to evaluate A and B.

32. **Extremising a function**: To find the extrema of a function $f(x)$, set $\dfrac{df(x)}{dx} = 0$.

The values of x that solve this equation are x_{ext}, the locations of the extrema. The function extrema are then given by $f(x_{\text{ext}})$.

The nature of each extremum is determined by the second derivative:

$$\left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_{\text{ext}}} \begin{cases} < 0, & \text{maximum;} \\ = 0, & \text{inflection point;} \\ > 0, & \text{minimum.} \end{cases}$$