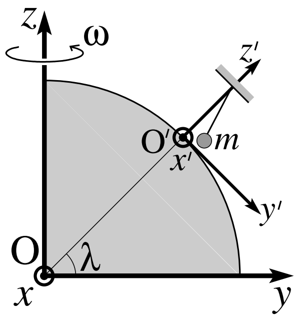
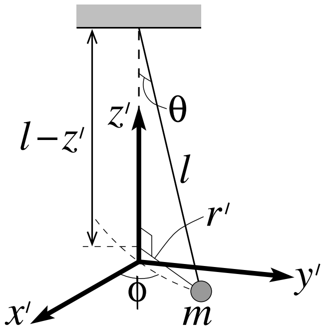
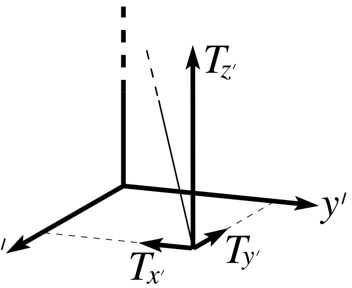
THE FOUCAULT PENDULUM

PHYS 2302, Saint Mary's University D. A. Clarke, October 2019

Consider a *Foucault pendulum* (Léon Foucault, 1819–1868), whose plane of rotation rotates freely about vertical axis passing through its support.







From equation (1) in §4.4 of the lecture notes, we have (ignoring centrifugal term $\propto \omega^2$):

$$m\ddot{\vec{r}}\,' = \vec{T} + m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}}\,', \tag{1}$$

The (x', y', z') components of $\vec{T}$ in terms of θ and ϕ are:

$$\vec{T} = T(-\sin\theta\cos\phi,\, -\sin\theta\sin\phi,\, \cos\theta),$$

where, from the figure above:

$$\cos\theta = \frac{l-z'}{l}; \quad \sin\theta = \frac{r'}{l}; \quad \cos\phi = \frac{x'}{r'}; \quad \sin\phi = \frac{y'}{r'}.$$

Thus,

$$\vec{T} = T\underbrace{\Big(-\frac{x'}{l},\, -\frac{y'}{l},\, \frac{l-z'}{l}\Big)}_{\textit{directional cosines}} = -\frac{x'}{l}T\hat{\imath}' - \frac{y'}{l}T\hat{\jmath}' + \frac{l-z'}{l}T\hat{k}'.$$

Next, $\vec{g} = -g\hat{k}'$ and, from equation (2) in §4.4 of the lecture notes, we have:

$$\vec{\omega}\times\dot{\vec{r}}\,' = \omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda)\hat{\imath}' + \omega\dot{x}'\sin\lambda\,\hat{\jmath}' - \dot{x}'\cos\lambda\,\hat{k}'.$$

Thus, the components of equation (1) are:

$$\ddot{x}' = -\frac{x'}{l}\frac{T}{m} + 2\dot{y}'\omega\sin\lambda - 2\dot{z}'\omega\cos\lambda; \tag{2}$$

$$\ddot{y}' = -\frac{y'}{l}\frac{T}{m} - 2\dot{x}'\omega\sin\lambda; \tag{3}$$

$$\ddot{z}' = \frac{l-z'}{l}\frac{T}{m} - g + 2\dot{x}'\omega\cos\lambda. \tag{4}$$

For small angles ($\theta \ll 1$), vertical speed $\dot{z}' \approx 0 \;\Rightarrow\; \ddot{z}' \approx 0,\; (l-z')/l \approx 1$.

Further, for $\dot{x}' \sim 0.1\,\mathrm{m/s}$, $2\dot{x}'\omega\cos\lambda \sim 10^{-5} \ll g \sim 10$.

Thus, equation (4) $\Rightarrow T/m \approx g$, and equations (2) and (3) become:

$$\ddot{x}' \approx -x'\frac{g}{l} + 2\dot{y}'\omega'; \tag{5}$$

$$\ddot{y}' \approx -y'\frac{g}{l} - 2\dot{x}'\omega', \tag{6}$$

where $\omega' \equiv \omega\sin\lambda$, component of $\vec{\omega}$ in $\hat{k}'$ direction.

Terms $\propto g/l$ make equations (5) and (6) harder to solve than projectile equations.

A "standard trick" is to add $\sqrt{-1} = i$ times equation (6) to equation (5) to get:

$$\begin{aligned} \ddot{x}' + i\ddot{y}' + (x' + iy')\frac{g}{l} &= 2\omega'(\dot{y}' - i\dot{x}') = -2i\omega'(\dot{x}' + i\dot{y}') \\ \Rightarrow \quad \ddot{q}' + 2i\omega'\dot{q}' + \Omega^2 q' &= 0, \end{aligned} \tag{7}$$

where $q' \equiv x' + iy'$ and $\Omega^2 \equiv g/l$.

This is just the damped harmonic oscillator equation with an imaginary damping constant, $2i\omega'$, whose solution is:

$$q'(t) = e^{-i\omega' t}\left(Ae^{i\sqrt{\Omega^2+\omega'^2}t} + Be^{-i\sqrt{\Omega^2+\omega'^2}t}\right), \tag{8}$$

where A and B are set from initial conditions (*e.g.*, equation 2 in §2.4 of the lecture notes).

Exercise: Verify by direct substitution that equation (8) solves equation (7).

Consequences of *imaginary* damping constant:

- renders oscillatory what was the exponential decay factor, $e^{-\omega' t} \to e^{-i\omega' t}$;
- renders argument under radical positive definite, $\Omega^2 - \omega'^2 \to \Omega^2 + \omega'^2$.

Now, $\omega'^2 \sim (7.29 \times 10^{-5})^2 \sim 5 \times 10^{-9} \ll \Omega^2 \sim 100 \;\Rightarrow$ equation (8) simplifies to:

$$\boxed{q'(t) \;=\; e^{-i\omega' t}\left(Ae^{i\Omega t} + Be^{-i\Omega t}\right).} \tag{9}$$

To interpret this solution, consider first an inertial frame in which $\omega' = 0$:

$$\begin{aligned}
(9) \quad \Rightarrow \quad q(t) \;&=\; Ae^{i\Omega t} + Be^{-i\Omega t} && (10)\\
&=\; A(\cos\Omega t + i\sin\Omega t) + B(\cos\Omega t - i\sin\Omega t)\\
&=\; (A+B)\cos\Omega t + i(A-B)\sin\Omega t \;=\; x(t) + iy(t)
\end{aligned}$$

$$\Rightarrow \quad x(t) \;=\; (A+B)\cos\Omega t; \quad y(t) \;=\; (A-B)\sin\Omega t,$$

equating real and imaginary parts. Thus,

$$\frac{x^2}{(A+B)^2} + \frac{y^2}{(A-B)^2} \;=\; 1,$$

an ellipse with semi-major (semi-minor) axis $A+B$ $(A-B)$.

Without dissipation, m traces over the same ellipse forever.

Back to the rotating frame, substitute equation (10) into equation (9):

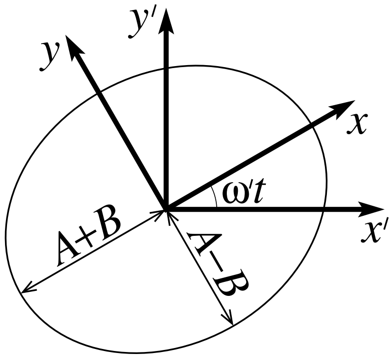
$$q'(t) \;=\; e^{-i\omega' t}q(t) \;=\; (\cos\omega' t - i\sin\omega' t)\big(x(t) + iy(t)\big)$$

$$= \cos\omega' t\, x(t) + \sin\omega' t\, y(t) + i\big(-\sin\omega' t\, x(t) + \cos\omega' t\, y(t)\big)$$

$$= x'(t) + iy'(t)$$

$$\Rightarrow \quad \begin{cases} x'(t) = \cos\omega' t\, x(t) + \sin\omega' t\, y(t) \\ y'(t) = -\sin\omega' t\, x(t) + \cos\omega' t\, y(t) \end{cases}$$

$$\Rightarrow \quad \boxed{\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \cos\omega' t & \sin\omega' t \\ -\sin\omega' t & \cos\omega' t \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.}$$



Path traced out by m in O′ is same as in O but rotated by angle $\omega' t$.

On earth, spherical pendulum traces out an ellipse (straight line for $A = B$) that precesses at frequency $\omega' = \omega \sin\lambda$.

Period of precession: $T = \dfrac{2\pi}{\omega\sin\lambda} = \dfrac{24 \text{ hr}}{\sin\lambda} = 33.9$ hr at $\lambda = 45°$.

YouTube videos for Foucault pendulum:

1. Animation of Foucault's pendulum;
2. Foucault's pendulum in the Pantheon, Paris.