Three lessons in solving ODEs

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1 First lesson

Ordinary differential equations (ODE) have *univariate* function derivatives:

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) \;=\; F.$$

Partial differential equations (PDE) have *multivariate* function derivatives:

$$-\frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \;=\; i\hbar\frac{\partial\psi(x,t)}{\partial t}.$$

Equation whose highest order derivative is n is an n^{th} *order ODE* (*PDE*).

- PHYS 2302/2303: simplest types of 1st and 2nd order ODEs
- MATH 2303: *any* 1st order ODE; 2nd order ODEs with constant coefficients
- PHYS 3200: *any* 2nd order ODE; *separable* 2nd order PDEs

Consider the two equations:

$$\frac{dy}{dx} \;=\; ay \qquad \text{and} \qquad x \;=\; \frac{1}{a}\int\frac{dy}{y}, \tag{1.1}$$

a first order ODE and an integral equation.

These are the same equation!

$$\frac{dy}{dx} = ay \quad\Rightarrow\quad \frac{dy}{y} = a\,dx \quad\Rightarrow\quad \int\frac{dy}{y} = \int a\,dx = ax \quad\Rightarrow\quad x = \frac{1}{a}\int\frac{dy}{y}.$$

To find $y(x)$, we "do the integral" and get:

$$x = \frac{1}{a}\ln y + c \quad \Rightarrow \quad \ln y = a(x-c) \quad \Rightarrow \quad y(x) = e^{a(x-c)} = Ce^{ax},$$

where $C = e^{-ac}$ is a constant set from a boundary condition. So, if $y = y_0$ at $x = 0$,

$$y(0) = Ce^{a(0)} = C = y_0 \quad \Rightarrow \quad y(x) = y_0 e^{ax},$$

is the final solution.

Both Eq. (1.1) are solved in "direction of antidifferentiation":

$$\xrightarrow{\text{direction of antidifferentiation (integration)}}$$

$$\cdots \quad \frac{d^2f(x)}{dx^2} \quad \frac{df(x)}{dx} \quad f(x) \quad \int^x f(y)dy \quad \int^x\!\!\int^y f(z)dzdy \quad \cdots$$

$$\xleftarrow[\text{direction of differentiation}]{}$$

$$\frac{dy}{dx} \to y(x) \qquad \text{and} \qquad g(y) = \frac{1}{y} \to \int g(y)dy.$$

And there's the rub! *There's no algebraic way to antidifferentiate*!

Direction of differentiation is different. Given $f(x)$, we can *always* use the definition of a derivative to find $f'(x)$ algebraically.

Example: $f(x) = \cos x$. Then,

$$\frac{df}{dx} = \lim_{\delta x \to 0} \frac{\cos(x+\delta x) - \cos x}{\delta x} = \lim_{\delta x \to 0} \frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x}{\delta x},$$

using a trig identity. Then, for $\delta x \to 0$, $\cos \delta x \to 1$ and $\sin \delta x \to \delta x$,

$$\Rightarrow \quad \frac{df}{dx} = \lim_{\delta x \to 0} \frac{\cancel{\cos x} - \sin x \,\cancel{\delta x} - \cancel{\cos x}}{\cancel{\delta x}} = -\sin x.$$

Example: $f(x) = \ln x$. Then,

$$\begin{aligned}\frac{df}{dx} &= \lim_{\delta x \to 0} \frac{\ln(x+\delta x) - \ln x}{\delta x} = \lim_{\delta x \to 0} \frac{\ln \big[x(1+\delta x/x)\big[- \ln x}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\cancel{\ln x} + \ln(1+\delta x/x) - \cancel{\ln x}}{\delta x} = \lim_{\delta x \to 0} \frac{\cancel{\delta x}/x}{\cancel{\delta x}} = \frac{1}{x},\end{aligned}$$

since $\ln(1+\epsilon) = \epsilon + \dfrac{\epsilon^2}{2} + \dfrac{\epsilon^3}{3} + \cdots \to \epsilon$ for $\epsilon \ll 1$.

One may need to know trig identities or expansions, but one can always find a derivative algebraically from the definition of a derivative.

Not so for antidifferentiation. Whether integrating or solving an ODE, there always comes a time when one "simply has to recognise the solution".

All methods of integration or solving ODEs boil down to beating the equation into a recognisable form, then writing down the answer.

In solving Eq. (1.1), we beat the ODE algebraically to:

$$x = \frac{1}{a}\int \frac{dy}{y},$$

at which point we "did the integral" by recognising it and writing down:

$$x = \frac{1}{a}\ln y + c,$$

then continued with the algebra.

In PHYS 2302, we shall limit ourselves to:

1^{st} order ODEs that can be:

- easily converted to an integral; or
- solved directly by "inspection",

2^{nd} order ODEs that can be:

- converted to two first order ODEs (then see above); or
- solved directly by "inspection"; or
- solved by trial exponential solutions.

Example: Solve $\dfrac{dy}{dx} = ay$ again, this time by "inspection".

Solution: For simple ODEs such as this, ask:

What function, $y(x)$, has a first derivative equal to itself times a constant?

Dropping "times a constant", the answer is easy: e^x is the only function whose derivative is itself:

$$\frac{de^x}{dx} = e^x.$$

So, if we let $y = e^x$, we'd have $\dfrac{dy}{dx} = y$; we want ay.

After further inspection, try $y = e^{ax}$ so that by the chain rule,

$$\frac{dy}{dx} = ae^{ax} = ay,$$

as desired.

But wait! Ce^{ax} (C any constant) still solves the ODE:

$$\frac{dy}{dx} = \frac{d}{dx}Ce^{ax} = C\frac{d}{dx}e^{ax} = Cae^{ax} = aCe^{ax} = ay.$$

Thus, $y(x) = Ce^{ax}$ is the general solution, as found before by integration.

Whether by integration or inspection, there's always an "alakazam moment" where something is simply recognised.

- integration may seem a little more elegant, but it's usually longer
- inspection may seem daunting, but if you can see it, go for it!

The method used to convert the ODE to an integral is called *separation of variables.* Starting with,

$$\frac{dy}{dx} = ay,$$

put all occurrences of y on one side of the equation, x on the other:

$$\frac{dy}{y} = a\,dx.$$

Once in this form, integrate both sides and solve.

Not all first order ODEs can be separated, *e.g.* $\dfrac{dy}{dx} = y + x$. (Try it!)

Those that can are said to be *separable.*

The general form for a separable first order ODE is:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)},$$

where $f(x)$ and $g(y)$ are arbitrary functions[1]. In this form, we write:

$$g(y)dy = f(x)dx \quad \Rightarrow \quad \int g(y)dy = \int f(x)dx,$$

then, "do the integrals", then solve for $y(x)$ algebraically.

For examples of solving first order ODEs by separation of variables, see practise problem 1 in assignment 2.

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[1] Note we could equally write down the general form as $\frac{dy}{dx} = f(x)h(y)$. It's entirely a matter of taste whether the RHS is written as a product or a quotient.

2 Second lesson

In the polynomial,

$$y(x) \;=\; a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \;=\; \sum_{i=0}^{\infty} a_i x^i,$$

where a_i are constants (no x-dependence),

- a_0 is the *constant* term,
- $a_1 x$ is the *linear* term ($y = a_1 x$ is a line!),
- $a_2 x^2$ is the *quadratic* term,
- $a_3 x^3$ is the *cubic* term, *etc.*

A polynomial ending $\left\{ \begin{array}{l} \text{at the linear term} \\ \text{beyond linear term} \end{array} \right\}$ is said to be $\left\{ \begin{array}{l} \textit{linear} \\ \textit{non-linear} \end{array} \right\}$.

Since functions like e^x, $\ln(1+x)$, $\cos x$, $\sin x$ all have *power-law expansions*:

$$\begin{aligned}
e^x \;&=\; 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \;=\; \sum_{i=0}^{\infty} \frac{x^i}{i!} \\
\ln(1+x) \;&=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \;=\; \sum_{i=1}^{\infty} \frac{(-1)^{i+1} x^i}{i} \\
\cos x \;&=\; 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \;=\; \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}; \\
\sin x \;&=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \;=\; \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!},
\end{aligned}$$

they are also said to be non-linear.

ODEs (and PDEs) can be *linear* and *non-linear* as well.

A *general* 2^{nd} order ODE has form:

$$\frac{d^2y}{dx^2} = y''(x) = f(y', y, x),$$

where f is *any* function of y', y, and x.

If $f(y', y, x)$ is a linear function of y' and y (but still may be non-linear in x), the ODE is *linear*. Otherwise, the ODE is *non-linear*.

Examples:

$$\begin{aligned}
y''(x) &= xe^{y'} + y\sin(y^2), && \text{non-linear ODE;}\\
y''(x) &= \frac{1}{y} + yy' + xy' + x^2, && \text{non-linear ODE;}\\
y''(x) &= -\frac{1}{x}y' + \left(\frac{n^2}{x^2} - 1\right)y, && \text{linear ODE; Bessel's equation;}\\
\ddot{x}(t) &= -\frac{c}{m}\dot{x} - \frac{k}{m}x, && \text{linear ODE; damped oscillator.}
\end{aligned}$$

Most ODEs/PDEs in physics are linear, but by no means all!

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = f(x), \quad \text{non-linear PDE; Euler's equation (fluid dynamics).}$$

A *general* linear 2^{nd} order ODE has form:

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = \begin{cases} 0, & \textit{homogeneous;} \\ d(x) \neq 0, & \textit{inhomogeneous,} \end{cases} \tag{2.1}$$

where a, b, c, d are known functions of x (perhaps constant).

Theorem 2.1. Principle of superposition. *If $y_1(x)$ and $y_2(x)$ both solve the homogeneous ODE (2.1), then so does a* linear combination *of the two:*

$$y_3(x) = Ay_1(x) + By_2(x), \tag{2.2}$$

where A, B are any *constants.*

Proof. If we substitute Eq. (2.2) into the LHS of Eq. (2.1) and get 0, then Eq. (2.2) solves the homogeneous Eq. (2.1). To that end,

$$\begin{aligned} ay_3'' + by_3' + cy_3 &= a(Ay_1'' + By_2'') + b(Ay_1' + By_2') + c(Ay_1 + By_2) \\ &= A(\underbrace{ay_1'' + by_1' + cy_1}_{0}) + B(\underbrace{ay_2'' + by_2' + cy_2}_{0}) = 0, \end{aligned}$$ ☐

Theorem 2.1 actually applies to any *linear* n^{th} order ODE.

Since A and B are arbitrary, Eq. (2.2) represents an ∞ of solutions to Eq. (2.1). However,

Theorem 2.2. (stated without proof): *A* 2^{nd} *order ODE has two* linearly independent *solutions (one not a multiple of the other) from which all other solutions can be constructed [e.g., by Eq. (2.2)].*

In general, an n^{th} order ODE has n linearly independent solutions.

For an example of solving a linear, second-order homogeneous ODE by "inspection", and then constructing is general solution, see practise problem 1 in assignment 3.

3 Third lesson

Consider the *inhomogeneous, linear* 2^{nd} *order ODE*:

$$ay''(x) + by'(x) + cy(x) = d(x), \tag{3.1}$$

where a, b, c are constant.

Let $y_{\text{p}}(x)$ be any function that solves Eq. (3.1):

- there are an infinity of solutions to any ODE, $y_{\mathrm{p}}(x)$ is just one that solves Eq. (3.1);
- $y_{\mathrm{p}}(x)$ is not necessarily the general solution to Eq. (3.1);
- $y_{\mathrm{p}}(x)$ is known as the *particular solution*.

Further, let $y_{\mathrm{h}}(x)$ be the *general solution* to the *homogeneous* version of Eq. (3.1):

$$ay''(x) + by'(x) + cy(x) \;=\; 0. \tag{3.2}$$

Thus, $y_{\mathrm{h}}(x)$ has the form:

$$y_{\mathrm{h}}(x) \;=\; Ay_1(x) + By_2(x),$$

where:

- A, B are free parameters (independent of x, set by boundary conditions);
- $y_1(x)$, $y_2(x)$ are *independent* functions (one not a multiple of the other) that each solve Eq. (3.2) individually.

Theorem 3.1. *The* general *solution to Eq. (3.1) is:*

$$y(x) \;=\; y_{\mathrm{h}}(x) + y_{\mathrm{p}}(x) \;=\; Ay_1(x) + By_2(x) + y_{\mathrm{p}}(x). \tag{3.3}$$

Proof. Substitute Eq. (3.3) into the LHS of Eq. (3.1) to get:

$$\begin{aligned} &a\big(Ay_1'' + By_2'' + y_{\mathrm{p}}''\big) + b\big(Ay_1' + By_2' + y_{\mathrm{p}}'\big) + c\big(Ay_1 + By_2 + y_{\mathrm{p}}\big) \\ &\qquad = A\big(\underbrace{ay_1'' + by_1' + cy_1}_{0;\text{ solves (3.2)}}\big) + B\big(\underbrace{ay_2'' + by_2' + cy_2}_{0;\text{ solves (3.2)}}\big) + \underbrace{ay_{\mathrm{p}}'' + by_{\mathrm{p}}' + cy_{\mathrm{p}}}_{d(x);\text{ solves (3.1)}} \\ &\qquad\quad = d(x) \;=\; \text{RHS of Eq. (3.1)}. \end{aligned}$$

Thus, Eq. (3.3) solves Eq. (3.1). It is the *general solution* because it has two free parameters, A and B, to be set by boundary conditions. □

Thus, the strategy for solving an inhomogeneous equation like Eq. (3.1) is:

1. Find the *general* solution to the homogeneous equation, (3.2), $y_{\mathrm{h}}(x)$.
2. Find *any* solution to Eq. (3.1), $y_{\mathrm{p}}(x)$, by:
   - inspection;
   - a clever substitution;
   - *variation of parameters* (PHYS 3200).
3. Construct the general solution to Eq. (3.1), $y(x) = y_{\mathrm{h}}(x) + y_{\mathrm{p}}(x)$.
4. Apply boundary conditions to evaluate the free parameters (constants of integration) A and B, thereby finding the *specific* solution.

For an example of solving a linear, second-order inhomogeneous ODE by finding the homogenous and particular solutions, and then applying boundary conditions to construct the specific solution, see practise problem 3 in assignment 4.