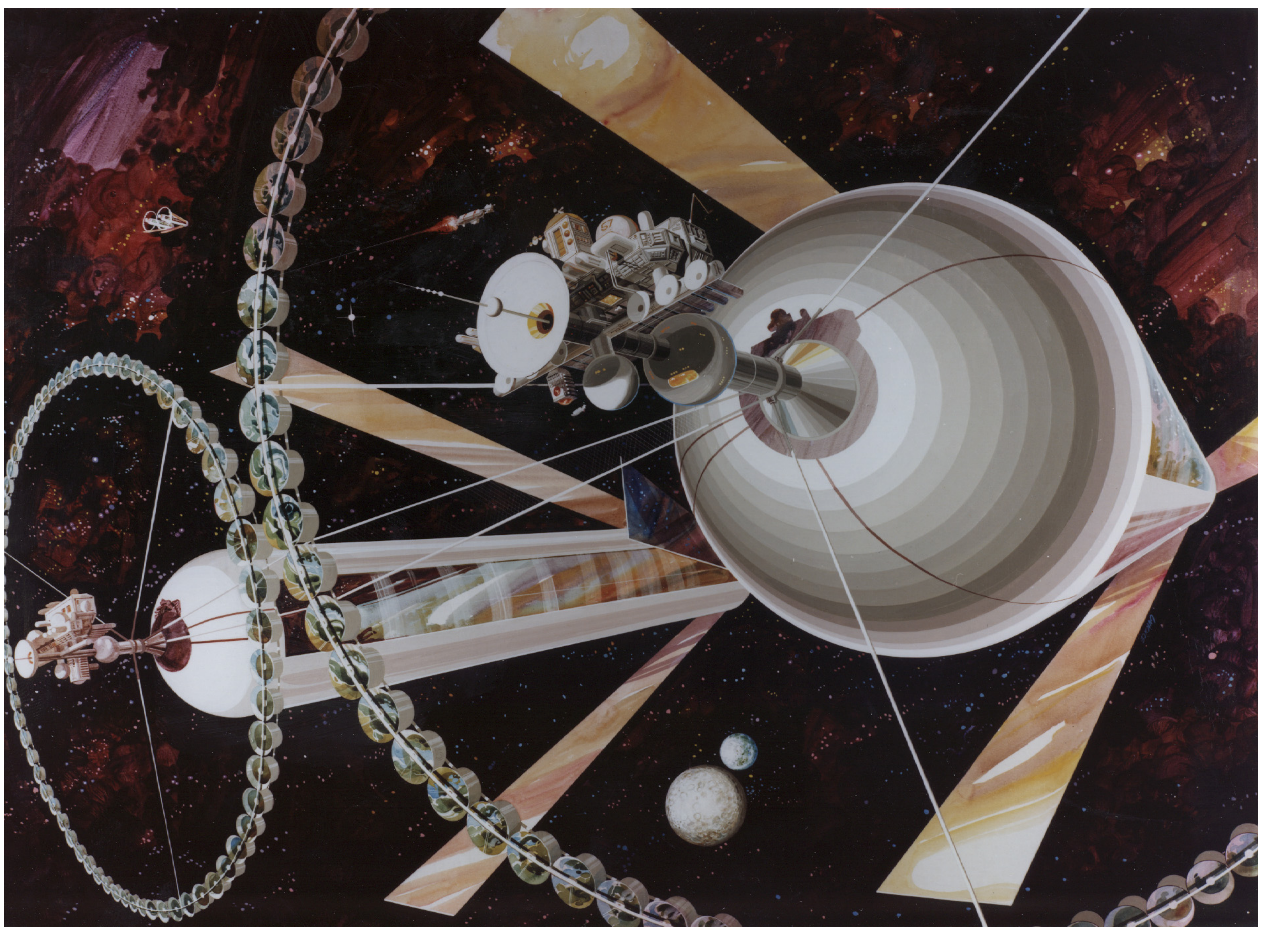
The “O’Neill cylinder”

PHYS 2302, Saint Mary’s University — D. A. Clarke, October 2019

As introduced in example 5.5.1 in the text (p. 207–212, ed. 7), the *O’Neill cylinder* is a hypothetical giant rotating cylinder in deep space capable of sustaining life—even entire civilisations—on its inner surface. Proposed and described by Gerard O’Neill in his *The High Frontier: Human Colonies in Space*, and featured in Arthur C. Clarke’s *Rendez-vous with Rama*, such an object with a radius $R \sim 100\,\mathrm{km}$, and angular speed $\omega \sim 10^{-2}\,\mathrm{rad\,s^{-1}}$ ($T \sim 10\,\mathrm{min}$), would provide an effective $g = \omega^2 R \sim 10\,\mathrm{m\,s^{-2}}$ for inhabitants living on the inside surface.

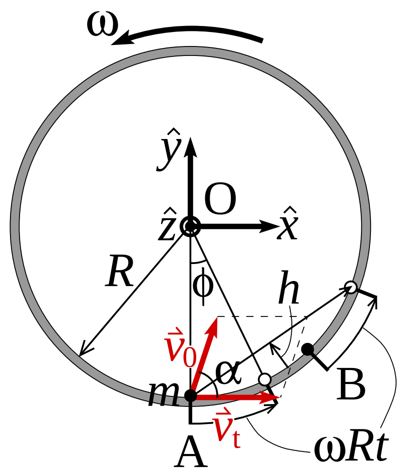
While still an object of science fiction, there is no reason why such a device couldn’t be constructed one day, and there is even less reason why we can’t use it now as a laboratory for dynamics in accelerating frames of reference!



Indeed, it seems that two of the factions in our hypothetical space colony don’t get along very well together, and are firing catapults at each other…

As shown in the figure, a large, closed, hollow cylinder of radius R rotates with angular speed ω about its axis, $\hat{z}$. The inhabitants at A are launching projectiles of mass m and launch velocity $\vec{v}_0$ at the inhabitants at B.

Let the tangential speed of the cylindrical surface be $v_{\rm t} = \omega R$. Then, relative to inertial frame, O,



- No real forces $\Rightarrow$ m has constant velocity:

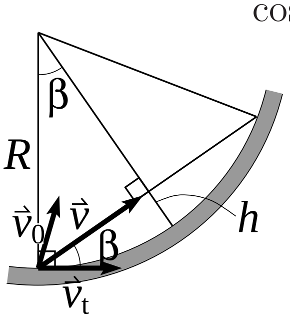
  $$\vec{v} \;=\; \vec{v}_0 + \vec{v}_{\rm t} \;=\; (v_0\cos\alpha + \omega R)\hat{x} + v_0 \sin\alpha\,\hat{y};$$

- In time t, A and B rotate by angle $\phi = \omega t$;

- In same time, m travels distance vt to hit B.

Problem 1: Find maximum "height", h, m gets from inner surface.

Solution: Let β be the angle between $\vec{v}$ and $\vec{v}_t$ when m is launched. Then,



$$\cos\beta \;=\; \frac{\vec{v}_{\rm t}\cdot\vec{v}}{|\vec{v}_{\rm t}||\vec{v}|} \;=\; \frac{\cancel{\omega R}(v_0\cos\alpha+\omega R)}{\cancel{\omega R}\sqrt{(v_0\cos\alpha+\omega R)^2+(v_0\sin\alpha)^2}}$$

$$= \frac{v_0\cos\alpha+\omega R}{\sqrt{v_0^2+2v_0\omega R\cos\alpha+\omega^2R^2}} \;=\; \frac{\cos\alpha+\xi}{\sqrt{1+2\xi\cos\alpha+\xi^2}},$$

where $\xi = \omega R/v_0$. Then, from the figure:

$$h \;=\; R(1-\cos\beta) \;=\; R\left(1-\frac{\cos\alpha+\xi}{\sqrt{1+2\xi\cos\alpha+\xi^2}}\right). \tag{1}$$

Problem 2: Derive equations of motion from rotating reference frame A.

Solution Step 1: "Hunting and gathering".

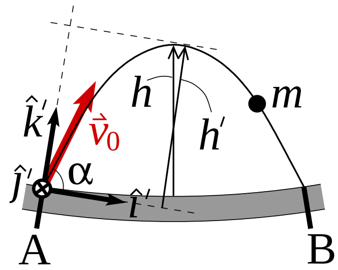
angular vel. of A relative to O: $\quad \vec{\omega} \;=\; \omega\hat{z} \;=\; -\omega\hat{\jmath}'$

position of m relative to A: $\vec{r}' = x'\hat{\imath}' + z'\hat{k}'$;

velocity of m relative to A: $\vec{v}' = \dot{x}'\hat{\imath}' + \dot{z}'\hat{k}'$;

acceleration of m relative to A: $\vec{a}' = \ddot{x}'\hat{\imath}' + \ddot{z}'\hat{k}'$;

acceleration of A relative to O: $\vec{A}_0 = \omega^2 R\hat{k}'$.



Step 2: Do the cross products.

$$\text{Coriolis:} \quad -2m\vec{\omega}\times\vec{v}' = 2m\omega\hat{\jmath}'\times(\dot{x}'\hat{\imath}' + \dot{z}'\hat{k}') = 2m\omega(-\dot{x}'\hat{k}' + \dot{z}'\hat{\imath}');$$

$$\begin{aligned}\text{centrifugal:} \quad -m\vec{\omega}\times(\vec{\omega}\times\vec{r}') &= -m\omega^2\hat{\jmath}'\times\big(\hat{\jmath}'\times(x'\hat{\imath}' + z'\hat{k}')\big)\\ &= -m\omega^2\hat{\jmath}'\times(-x'\hat{k}' + z'\hat{\imath}') = m\omega^2(x'\hat{\imath}' + z'\hat{k}') = m\omega^2\vec{r}'.\end{aligned}$$

Step 3: Assemble the forces (equation 1, §3.3):

$$\begin{aligned}\vec{F}' = m\vec{a}' &= \cancelto{0}{\vec{F}} - m\cancelto{0}{\dot{\vec{\omega}}}\times\vec{r}' - 2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') - m\vec{A}_0\\ \Rightarrow \quad \ddot{x}'\hat{\imath}' + \ddot{z}'\hat{k}' &= 2\omega(-\dot{x}'\hat{k}' + \dot{z}'\hat{\imath}') + \omega^2(x'\hat{\imath}' + z'\hat{k}') - \omega^2 R\hat{k}'\\ \Rightarrow \quad \ddot{x}' = 2\omega\dot{z}' + \omega^2 x' \quad &\text{and} \quad \ddot{z}' = -2\omega\dot{x}' + \omega^2(z' - R). \qquad (2)\end{aligned}$$

In the limit $R \gg z', x'$ and $\omega R \gg \dot{x}', \dot{z}'$, equations (2) reduce to:

$$\ddot{x}' \approx 0 \quad \text{and} \quad \ddot{z}' \approx -\omega^2 R \sim -g,$$

and we have the same ballistics problem as we do near the earth's surface.

Problem 3: Solve equations (2) for $z(t)$ and $x(t)$.

Solution: Let $u = z' + ix'$ where $i = \sqrt{-1}$. Then, from equations (2):

$$\begin{aligned}\ddot{u} &= \ddot{z}' + i\ddot{x}' = -2\omega\dot{x}' + \omega^2 z' - \omega^2 R + i(2\omega\dot{z}' + \omega^2 x')\\ &= \underbrace{2\omega(-\dot{x} + i\dot{z})}_{2\omega i(\dot{z}+i\dot{x})} + \omega^2(z' + ix') - \omega^2 R = 2\omega i\dot{u} + \omega^2 u - \omega^2 R\end{aligned}$$

$$\Rightarrow \quad \ddot{u} - 2i\omega\dot{u} - \omega^2 u = -\omega^2 R \tag{3}$$

To solve (3), first consider the homogeneous equation:

$$\ddot{u}_{\mathrm{h}} - 2i\omega\dot{u}_{\mathrm{h}} - \omega^2 u_{\mathrm{h}} = 0,$$

for which $e^{i\omega t}$ and $te^{i\omega t}$ are solutions. Thus, general homogeneous solution is:

$$u_{\mathrm{h}}(t) = ae^{i\omega t} + bte^{i\omega t}, \quad a, b \in \mathbb{C}.$$

A particular solution is $u_{\mathrm{p}} = R$, and thus general solution to (3) is:

$$u(t) = ae^{i\omega t} + bte^{i\omega t} + R. \tag{4}$$

To evaluate a and b, apply initial conditions. Thus, at $t = 0$,

$$\begin{aligned}
(x, z) &= (0, 0) & \Rightarrow \qquad u(0) &= 0; \\
(\dot{x}, \dot{z}) &= v_0(\cos\alpha, \sin\alpha) & \Rightarrow \qquad \dot{u}(0) &= v_0(i\cos\alpha + \sin\alpha) \\
& & &= iv_0(\cos\alpha - i\sin\alpha) = iv_0 e^{-i\alpha}.
\end{aligned}$$

Substituting these into (4), we get:

$$\begin{aligned}
0 &= a + R & \Rightarrow \quad a &= -R; \\
iv_0 e^{-i\alpha} &= i\omega a + b & \Rightarrow \quad b &= i(v_0 e^{-i\alpha} + \omega R)
\end{aligned}$$

Thus, (4) becomes:

$$\begin{aligned}
u(t) &= -Re^{i\omega t} + i(v_0 e^{-i\alpha} + \omega R)te^{i\omega t} + R \\
&= (-R + itv_0\cos\alpha + v_0 t\sin\alpha + it\omega R)e^{i\omega t} + R \\
&= \big((v_0 t\sin\alpha - R) + it(v_0\cos\alpha + \omega R)\big)\big(\cos\omega t + i\sin\omega t\big) + R \\
&= R + (v_0 t\sin\alpha - R)\cos\omega t - (v_0\cos\alpha + \omega R)t\sin\omega t \\
&\qquad + i\big((v_0 t\sin\alpha - R)\sin\omega t + (v_0\cos\alpha + \omega R)t\cos\omega t\big)
\end{aligned}$$

$$= z'(t) + ix'(t)$$

$$\Rightarrow \quad \boxed{\begin{aligned} z'(t) &= R(1 - \cos\omega t - \omega t \sin\omega t) - v_0 t \sin(\omega t - \alpha); \\ x'(t) &= R(\omega t \cos\omega t - \sin\omega t) + v_0 t \cos(\omega t - \alpha). \end{aligned}} \qquad (5)$$

Exercise: Confirm by direct substitution that equations (5) solve (2).

The approximation that $R \gg x', z'$ is equivalent to $\omega t \ll 1$. In this limit,

$$z'(t) \approx R\left(1 - 1 + \frac{(\omega t)^2}{2} + \cdots - (\omega t)^2 + \cdots\right) + v_0 t \sin\alpha \approx v_{0z} t - \frac{1}{2}(\omega^2 R)t^2$$

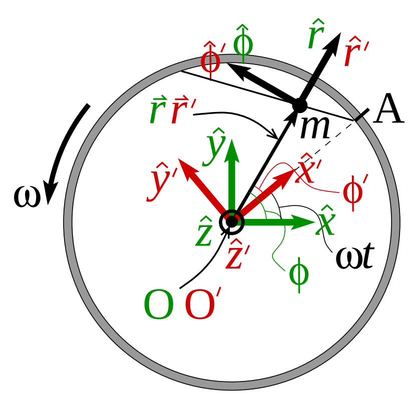
$$x'(t) \approx R(\omega t + \cdots - \omega t + \cdots) + v_0 t \cos\alpha \approx v_{0x} t,$$

ignoring terms $\mathcal{O}(\omega t)^3$. With $\omega^2 R = g$, these are the trajectory equations for a projectile near the earth's surface.

To find the "height", h', of m at the top of its trajectory as observed by A,

- set $dz'/dt = 0$ (first of equations 5), find t_{top}, solve for $h' = z'(t_{\text{top}})$.
- Solution must be obtained numerically (a root finder).
- Can see from the figure (top of page 3) that $h' > h$ (equation 1) and t_{top} is a bit more than half way through the trajectory.

Problem 4: Find equations of motion of projectile using coordinates rotating with cylinder, where origin O′ coincides with O, $\hat{z}'$ lies along cylinder rotation axis, and $\hat{x}'$ always points at A.



Solution: While we've defined Cartesian coordinates $(\hat{x}', \hat{y}', \hat{z}')$ to keep track of A from O′, we'll use cylindrical coordinates $(\hat{\imath}', \hat{\jmath}', \hat{k}')$ to track m. Note that $\hat{k} = \hat{z}$, $\hat{k}' = \hat{z}'$.

As shown in the figure, O and O$'$ attach the same cylindrical unit vectors to m. The only difference between O$'$ and O is $\phi' = \phi - \omega t$.

Step 1: "Hunting and gathering".

angular vel. of O$'$ relative to O: $\vec{\omega} = \omega \hat{z}'$

position of m relative to O$'$: $\vec{r}' = r'\hat{r}' + z'\hat{z}'$;

velocity of m relative to O$'$: $\vec{v}' = \dot{r}'\hat{r}' + r'\dot{\phi}'\hat{\phi}' + \dot{z}'\hat{z}'$;

acceleration of m relative to O$'$: $\vec{a}' = (\ddot{r}' - r'\dot{\phi}'^2)\hat{r}' + (2\dot{r}'\dot{\phi}' + r'\ddot{\phi}')\hat{\phi}' + \ddot{z}'\hat{z}'$;

acceleration of m relative to O: $\vec{a} = 0$ (no real forces);

acceleration of O$'$ relative to O: $\vec{A} = 0$ (O$'$ and O are coincident),

where $\vec{v}'$ and $\vec{a}'$ come from eqs. (1.12.2) and (1.12.3) in chapter 1 of ed. 7.

Step 2: Do the cross products.

$$\vec{\omega} \times \vec{v}' = \omega \hat{z}' \times (\dot{r}'\hat{r}' + r'\dot{\phi}'\hat{\phi}' + \dot{z}'\hat{z}') = \omega \dot{r}'\hat{\phi}' - \omega r'\dot{\phi}'\hat{r}',$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}') = \omega^2 \hat{z}' \times \left(\hat{z}' \times (r'\hat{r}' + z'\hat{z}')\right) = -\omega^2 r'\hat{r}'.$$

Step 3: Assemble the accelerations (third of equations 4, §3.2).

$$\vec{a}' = \cancelto{0}{\vec{a}} - \cancelto{0}{\dot{\vec{\omega}}} \times \vec{r}' - 2\vec{\omega} \times \vec{v}' - \vec{\omega} \times (\vec{\omega} \times \vec{r}') - \cancelto{0}{\vec{A}_0}$$

$$\Rightarrow \quad (\ddot{r}' - r'\dot{\phi}'^2)\hat{r}' + (2\dot{r}'\dot{\phi}' + r'\ddot{\phi}')\hat{\phi}' + \ddot{z}'\hat{z}' = -2(\omega \dot{r}'\hat{\phi}' - \omega r'\dot{\phi}'\hat{r}') + \omega^2 r'\hat{r}'.$$

Breaking this up into components, we get:

$$\ddot{r}' = r'\dot{\phi}'^2 + 2\omega r'\dot{\phi}' + r'\omega^2 \quad \Rightarrow \quad \boxed{\ddot{r}' = r'(\omega + \dot{\phi}')^2;}$$

$$r'\ddot{\phi}' = -2\dot{r}'\dot{\phi}' - 2\omega \dot{r}' \quad \Rightarrow \quad \boxed{\ddot{\phi}' = -\frac{2\dot{r}'}{r'}(\omega + \dot{\phi}');}$$

$$\text{and} \quad \boxed{\ddot{z}' = 0.}$$