Assignment 3, PHYS 2302

assigned Thursday, October 6; due Thursday, October 13

Problem 1 Consider the following linear, second-order, homogeneous ODE:

$$y''(x) - 4y'(x) = 0. \tag{1}$$

a) Find "by inspection" two linearly independent solutions to equation (1).

b) From your two linearly independent solutions, write down the general solution.

c) Show that when the boundary conditions $y(0) = -1$ and $y'(0) = -2$ are applied to your general solution in part b, you get:

$$y(x) = -e^{2x} \cosh 2x,$$

where $\cosh z \equiv \frac{1}{2}(e^z + e^{-z})$ is the *hyperbolic cosine function*, which we'll meet in an upcoming class.

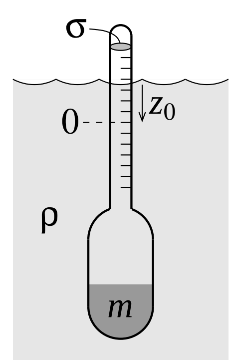
Problem 2 (FC 3.2 & 3.3)

a) A piston executes simple harmonic motion with an amplitude of $0.100\,\mathrm{m}$. If it passes through the centre of its motion with a speed $0.500\,\mathrm{m\,s^{-1}}$, find the period of oscillation.

b) A particle undergoes simple harmonic motion at a frequency of $20.0\,\mathrm{Hz}$. Find the displacement at any time, t, for the initial conditions $(t = 0)$ $x(0) = 0.100\,\mathrm{m}$ and $\dot{x}(0) = 0.250\,\mathrm{m\,s^{-1}}$.

Problem 3 This problem is based on *Archimedes' principle*, which states that the *buoyancy force* acting upward on an object floating or immersed in a liquid is equal to the weight of the displaced liquid.

An hydrometer is a device used to measure the density of a liquid based Archimedes' principle. It consists of a sealed, graduated hollow glass tube of uniform cross section, σ, and a ballast attached to the bottom for stability.



An hydrometer of mass m is lowered into a liquid of density ρ so that it floats freely. It is then depressed a distance z_0 and released.

a) Ignoring the viscosity of the liquid, show that the hydrometer oscillates as a simple harmonic oscillator with a period given by,

$$T = 2\pi\sqrt{\frac{m}{\rho\sigma g}}.$$

For the experimentalists, the liquid density can be calculated with:

$$\rho = \frac{4\pi^2 m}{T^2 \sigma g},$$

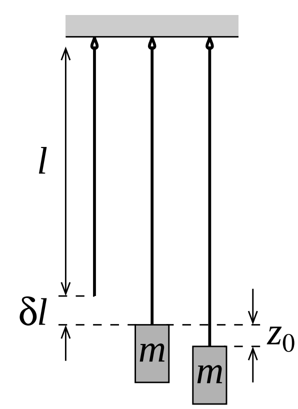
where T can be measured with a stopwatch.

b) Given the generic equation of motion for an SHO, $m\ddot{z} + kz = 0$, what is the effective spring constant, k, of the hydrometer?

Hint: Our discussion on vertical springs in §2.2 of the lecture notes shows that all forces establishing the equilibrium position of the system can be ignored in the FBD.

Problem 4 In this problem, you will learn about the elasticity of some materials, and an important example of simple harmonic motion not discussed in class. One might argue this falls under the realm of materials science or even engineering, but any good physicist should at least be aware of what is known as *Young's modulus*.

Consider a wire of length l, cross sectional area A, and negligible mass hanging vertically from a fixed anchor. If a mass m is hung from the free end, the wire stretches by an amount δl, as shown in the inset. Typically, $\delta l \ll l$ (*e.g.*, a thin metal wire will break before δl gets too big), although for a rubber band, δl could be comparable to and even greater than l.



Define the *strain*, Σ (the Greek capital 'S'), as the *distortion* of the wire caused by the weight of the hanging mass, m:

$$\Sigma \equiv \frac{\delta l}{l}, \tag{1}$$

a unitless quantity. Next, define the *stress* on the wire, S, as the *applied force per unit area* on the wire:

$$S \equiv \frac{F}{A}, \tag{2}$$

where $F = mg$ in this case. Thus, S has the units of *pressure*.

So far, everything has been definitions. Here's the only bit of physics: Experimentally, for *elastic* materials (which includes steel, by the way), *the stress is proportional to the strain*:

$$S \propto \Sigma \quad \Rightarrow \quad S = -Y\Sigma, \tag{3}$$

where the proportionality constant, $Y > 0$, is *Young's modulus* (units $\mathrm{N\,m^{-2}}$), a property of the material making up the wire (like its density or electrical conductivity). The negative sign means the stress and strain act in opposite directions. In the present example, the distortion is downward while the restoring force (the tension in the wire) acts upward.

Note that equation (3) is an approximation for how elastic materials actually behave, with the approximation better for smaller $\delta l/l$. Young's moduli for several common materials are given in the table below in units of GPa (1 gigaPascal $= 10^9\,\mathrm{N\,m^{-2}}$).

rubber band	0.01	glass	70	copper	117
aluminum	69	brass	110	steel	200

a) If m is pulled down an additional distance $z_0 \ll l$ then released, show that in the absence of any dissipative (frictional) forces, m moves up and down as a simple harmonic oscillator with a period given by:

$$T = 2\pi\sqrt{\frac{ml}{YA}}. \tag{4}$$

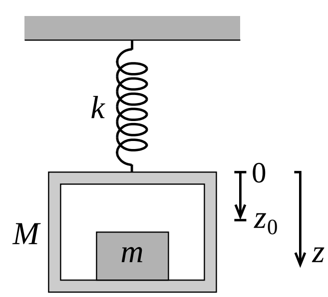
You may assume $\delta l \ll l$ and thus $l + \delta l \approx l$.

Hint: Review what we did in class for vertical oscillators (§2.2 in the class notes).

b) Find a numerical value for the period of oscillation for $m = 1.00\,\mathrm{kg}$ hanging on a length $l = 1.00\,\mathrm{m}$ of 12-gauge copper wire (2.05 mm in diameter). Why do you suppose this would be difficult to demonstrate in class?

c) By considering the equilibrium state of m hanging on the wire, show that the period given in equation (4) is the same as that of a simple pendulum of length δl.

Problem 5 (FC 3.8) A spring with spring constant k supports a box of mass M from the ceiling. Inside the box rests a block of mass m. The system is pulled down a distance z_0 from the equilibrium position and then released.



a) Find the normal force exerted by the bottom of the box on the block as a function of time.

Hint: Review the discussion on vertical springs in §2.2 of the class notes.

b) For what value of z_0 does the block just begin to leave the bottom of the box at the top of the oscillation?