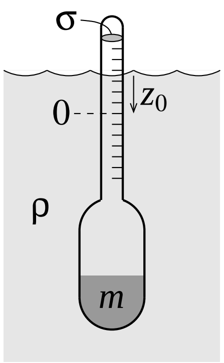
Assignment 4, PHYS 2302

assigned Thursday, October 13; due Thursday, October 27 (two weeks)

Problem 1 Consider the hydrometer problem of the previous assignment.

A hydrometer of mass m and cross section σ is placed carefully into a liquid of density ρ, then set into oscillation with initial amplitude z_0. It is noted that after 12 complete oscillations, the amplitude of oscillation has fallen by a factor of 2.



From equation (2.4.1) in the class notes, the equation of motion for a damped harmonic oscillator is: $\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = 0$. Using the result of the previous assignment, namely,

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{\rho\sigma g}}, \tag{1}$$

find the effective "damping coefficient", γ, for the oscillating hydrometer.

Problem 2 Starting from the general solution for the damped harmonic oscillator derived in class [equation (2.4.2) in the class notes], namely,

$$x(t) = e^{-\gamma t}\left(Ae^{qt} + Be^{-qt}\right), \tag{1}$$

where, as defined in class, $\gamma = b/2m$ is the damping coefficient, $q = \sqrt{\gamma^2 - \omega_0^2}$, and $\omega_0 = \sqrt{k/m}$ is the oscillator frequency when $b = 0$, use the initial conditions,

$$x(0) = 0 \qquad \text{and} \qquad \dot{x}(0) = v_0,$$

to show that,

$$x(t) = v_0 \begin{cases} e^{-\gamma t}\,\dfrac{\sinh qt}{q}, & \gamma > \omega_0 \text{ (overdamped)}; \\ te^{-\omega_0 t}, & \gamma = \omega_0 \text{ (critically damped)}; \\ e^{-\gamma t}\,\dfrac{\sin \omega_\text{d} t}{\omega_\text{d}}, & \gamma < \omega_0 \text{ (underdamped)}; \\ \dfrac{\sin \omega_0 t}{\omega_0}, & \gamma = 0 \text{ (undamped)}. \end{cases}$$

These initial conditions are tantamount to giving the oscillator a sharp blow (impulse) from the equilibrium point of the spring at $t = 0$, resulting in an initial velocity of v_0.

Problem 3 (FC 3.13) Suppose the amplitude of an oscillator drops off by a factor of e^{-1} of its initial value after n cycles.

a) Show that the ratio of the period of oscillation to the period of the same oscillator with no damping is given by:

$$\frac{T_{\mathrm{d}}}{T_0} = \sqrt{1 + \frac{1}{4\pi^2 n^2}}.$$

b) What is the quality factor, Q_{d}?

Hint: You may use the result of problem 3.9, namely that the ratio of successive maxima of a damped oscillator is $e^{-\gamma T_{\mathrm{d}}}$, where $T_{\mathrm{d}} = 2\pi/\omega_{\mathrm{d}}$ is the period of oscillation.

Problem 4 According to electromagnetic theory, an accelerated electron radiates energy at the rate of,

$$\dot{E} = -\frac{e^2 a^2}{6\pi\epsilon_0 c^3}, \tag{1}$$

where e is the electron charge, $a = \ddot{x}$ is the electron's instantaneous acceleration, ϵ_0 is the permittivity of free space, and c is the speed of light.

a) If an electron oscillates along a straight line with an angular frequency ω_0 and an amplitude A, show that the energy it radiates in one cycle is given by:

$$\Delta E = -\frac{e^2 A^2 \omega_0^3}{6\epsilon_0 c^3}.$$

You may assume the electron to be weakly damped. Thus, $\omega_{\mathrm{d}} \sim \omega_0$ and its motion is adequately described by $x(t) = A\cos(\omega_0 t)$ during a single cycle.

b) Show that the quality factor, Q_{d}, for the oscillating electron is given by,

$$Q_{\mathrm{d}} = \frac{3m_{\mathrm{e}}\epsilon_0 c^3}{e^2 f},$$

where m_{e} is the mass of the electron and $f = \omega_0/2\pi$ is the frequency in Hz.

c) If the oscillating electron is emitting visible light with wavelength $\lambda = 5{,}000\,\text{Å}$ ($1\,\text{Å} = 10^{-10}\,\mathrm{m}$), find a numerical value for Q_{d}.

d) How many periods of oscillation does electron undergo by the time its energy has fallen to $1/e$ of its original value? Was the initial assumption of weak dampening justified?

Problem 5 Solve by "inspection", "direct integration", and/or "trial exponentials" the linear, second-order, inhomogeneous ODE:

$$y''(x) - y'(x) - 2y(x) = f(x), \tag{1}$$

for boundary conditions $y(0) = 0$ and $y'(0) = 2$, where:

a) $f(x) = 1$;

b) $f(x) = -2e^{-2x}$.

As discussed in §2.5 of the course notes, to solve such an *inhomogeneous* [$f(x) \neq 0$] equation with boundary conditions, you must:

1. find two linearly independent solutions to the *homogeneous* equation [with $f(x) = 0$]; call these $y_1(x)$ and $y_2(x)$;

2. construct the general solution to the homogeneous equation,

$$y_{\mathrm{h}}(x) \;=\; Ay_1(x) + By_2(x),$$

   where A and B are constants;

3. by inspection or trial exponentials, find a *particular* solution, $y_{\mathrm{p}}(x)$ [*anything* that solves equation (1)];

4. write down the general solution to equation (1),

$$y(x) \;=\; y_{\mathrm{h}}(x) + y_{\mathrm{p}}(x);$$

5. and finally, apply boundary conditions to evaluate A and B.