Assignment 5, PHYS 2302

assigned Thursday, October 27; due Thursday, November 3

Problem 1 (FC 3.10 modified) A damped harmonic oscillator with $m = 5.0\,\mathrm{kg}$, $k = 180.\,\mathrm{N\,m^{-1}}$, and $b = 30.0\,\mathrm{kg\,s^{-1}}$ is subject to a driving force $F_0 \cos\omega t$, where $F_0 = 25.0\,\mathrm{N}$.

a) What value of ω results in steady-state oscillations with maximum amplitude (in the asymptotic limit)?

b) For the value of ω found in part a, what is the maximum amplitude and the corresponding phase shift?

Problem 2 (FC 3.18) Find the particular solution to the differential equation of motion for the driven, damped harmonic oscillator,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{-\alpha t} \cos\omega t, \tag{1}$$

and show that the phase and amplitude of the steady-state oscillations are given by:

$$\tan\phi = \frac{2\omega(\gamma - \alpha)}{\alpha^2 - \omega^2 - 2\gamma\alpha + \omega_0^2};$$

$$A = \frac{F_0/m}{\sqrt{\left(\alpha^2 - \omega^2 - 2\gamma\alpha + \omega_0^2\right)^2 + 4\omega^2(\gamma - \alpha)^2}}.$$

Hint: This is the same driving force used in class (equation 2.6.1 in the class notes), except for the factor $e^{-\alpha t}$. So how to proceed?

Well, from Euler's formula (equation 2.4.5 in the class notes),

$$e^{i\omega t} = \cos\omega t + i\sin\omega t,$$

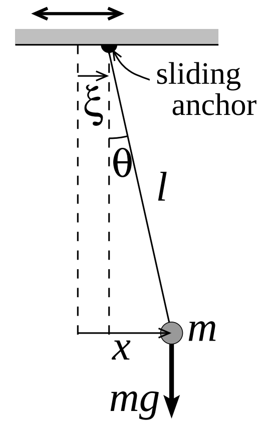
and thus $\cos\omega t$ is the *real part* of $e^{i\omega t}$ [written as $\cos\omega t = \Re(e^{i\omega t})$] and $\sin\omega t$ is the *imaginary part* of $e^{i\omega t}$ [written as $\sin\omega t = \Im(e^{i\omega t})$]. Thus, $e^{-\alpha t}\cos\omega t = \Re(e^{(-\alpha + i\omega)t}) = \Re(e^{\beta t})$, where $\beta \equiv -\alpha + i\omega \in \mathbb{C}$, Therefore, for the particular solution, try assuming the form,

$$x_\mathrm{p}(t) = A e^{\beta t - i\phi}, \tag{2}$$

rather than $A\cos(\omega t - \phi)$ as we did in class (equation 2.6.3), with the understanding that your actual solution will just be the real part of this.

So with that lead, plug equation (2) into equation (1), and see what happens…

Problem 3 A simple pendulum of length l and bob mass m swings back and forth with an effective weak damping coefficient, γ. The anchor can be driven to slide back and forth horizontally in simple harmonic motion with a maximum amplitude ξ_0.



Parts a, b, and c were done in the tutorial. You may use these results to answer the next three parts.

d) Find numerical values for A/ξ_0 and ϕ when the system is driven at:

$$i)\ \omega = \frac{\omega_0}{2}; \qquad ii)\ \omega = 2\omega_0.$$

Note that you don't need to know g or l and thus ω_0!

For phases, choose values for ϕ such that $0 < \phi < \pi$ rad. For what driving frequency, ω, is the displacement of the bob in phase and out of phase with the driver?

e) Write down expressions for the resonant frequency, ω_r, and the amplitude at resonance, A_{max}, and from these, find numerical values to three significant figures for $1 - \omega_r/\omega_0$ and A_{max}/ξ_0. Again, you don't need to know ω_0 for either of these values!

If the small angle approximation is to remain valid even at resonance (and thus $A_{max} < 0.1l$, say), what is the maximum permitted value of ξ_0/l?

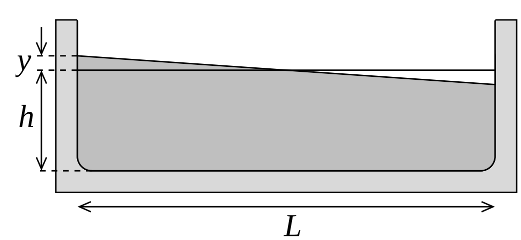
f) What is the phase, ϕ_r, at resonance? Now how would you say the phase of the bob relates to the driver?

Problem 4 (FC 3.11) A mass m moves along the x-axis subject to a restoring force $F_r = -\frac{17}{2}\beta^2 mx$ and a drag force $F_d = -3\beta m\dot{x}$, where x is the distance from the origin and β is a constant. In addition, a driving force, $F = mA\cos\omega t$ is applied to the particle along the x-axis, where A is another constant.

a) What value of ω results in steady-state oscillations about the origin ($x = 0$) with maximum amplitude?

b) For the value of ω found in part a, what is the maximum amplitude?

Problem 5 As depicted in the figure below, a bathtub of length L is filled with water to a depth h.



a) If the water is driven to slosh back and forth so that the water surface remains planar, show that the potential energy of the bathwater, U, is given by:

$$U(y) = \frac{\rho L w g}{6} y^2,$$

where ρ is the density of water, w is the width of the tub, and y is the additional depth of water above h at the left end of the tub ($x = 0$).

b) If you choose to do the bonus part d below, you'll find that the kinetic energy, K, of the horizontal motion of the water is given by,

$$K(\dot{y}) = \frac{\rho w L^3}{60h}\dot{y}^2.$$

Assuming the oscillation is weakly damped, set $E = U + K =$ constant, and show that the water level rises and falls as a simple harmonic oscillator. Thus, find its natural frequency, ω_0, and period of oscillation, T_0.

For small damping, $\omega_\mathrm{r} \sim \omega_\mathrm{d} \sim \omega_0$. Therefore, what is the resonant period of oscillation for a tub with $L = 1.5\,\mathrm{m}$ and $h = 0.3\,\mathrm{m}$? Does this jibe with your intuition? (Surely *every* kid discovered that sliding back and forth along the bottom of the tub with just the "right" frequency will spill water onto the floor!)

c) The Bay of Fundy is about 250 km long. If we model it as a big bathtub sloshing back and forth as it is driven by solar and lunar tidal forces, the bay itself would represent half of the tub with the other half stuck in the Gulf of Maine. Thus, take $L \sim 500\,\mathrm{km}$ and use $h \sim 50\,\mathrm{m}$ as its effective depth. What is the resonant period of oscillation for the Bay of Fundy? Compare this with the driving frequency of the moon (two high tides per 24.9 hr, the time for the moon to return to the same spot in the sky), and in one sentence, explain why the Fundy tides are so notoriously high.

d) (5 bonus points) Show that the kinetic energy, K, of the horizontal motion of the water is given by:

$$K(\dot{y}) = \frac{\rho w L^3}{60h}\dot{y}^2.$$

Here, we assume the amplitude of oscillation is sufficiently small that only the back-and-forth motion of the water contributes to K, with the vertical motion contributing a negligible amount. And yet note that K as given is as a function of $\dot{y}$. Hmmm...

Hint: While part a should be a 3 or 4 liner, calculating K is a bit trickier and took me a full page to do so. The way I approached it was to calculate $v(x)$ (horizontal motion of the water) in terms of x, the horizontal position along the tub where $0 \le x \le L$, and $\dot{y}$, the rate at which the water level rises and falls at the left end of the tub. For this, I had to invoke the idea that the volume of water entering a fixed rectangle at location x of height h and width dx is the same as that leaving. Doing this, you should find $dv/dx = -\dot{z}/h$, where $\dot{z}$ is the speed at which water is rising or falling at point x. This is a tricky problem, so don't feel badly if you don't get it!