Conservative forces

PHYS 2302, Saint Mary's University D. A. Clarke, July 2019

Theorem 3.1 from the class notes: If $\vec{F} = \vec{F}(\vec{r})$, then

$$\nabla \times \vec{F} = 0 \iff \exists\, U(\vec{r}) \;\Big|\; \vec{F} = -\nabla U.$$

The RHS reads: *There exists* ($\exists$) *a function,* $U(\vec{r})$*, such that* ($|$) $\vec{F} = -\nabla U$.

Theorems with an "if and only if" symbol ($\iff$) require both "directions" to be proven separately.

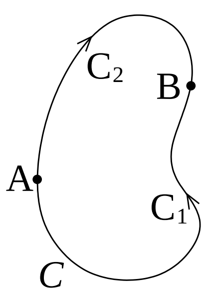
$\Leftarrow$ *proof*: Here, we assume $\vec{F} = -\nabla U$, and prove $\nabla \times \vec{F} = 0$. Thus,

$$\nabla \times \vec{F} = -\nabla \times (\nabla U) = 0,$$

using identity (3.1.5) in the class notes. □

$\Rightarrow$ *proof*: Here, we assume $\nabla \times \vec{F} = 0$, and prove $\vec{F} = -\nabla U$ for some $U(\vec{r})$.

Since $\nabla \times \vec{F} = 0$, we have from Stokes Theorem (page 3):



$$\int_S \nabla \times \vec{F} \cdot d\vec{\sigma} = \oint_C \vec{F} \cdot d\vec{r} = 0,$$

where $\int_S$ is an integral over an *open surface*[1] and $\oint_C$ is a *contour integral* over the loop that is the open surface's edge.

Now, break up the contour integral into two *path integrals*:

$$\oint_C \vec{F} \cdot d\vec{r} = \underbrace{\int_{\mathrm{A}}^{\mathrm{B}} \vec{F} \cdot d\vec{r} + \int_{\mathrm{B}}^{\mathrm{A}} \vec{F} \cdot d\vec{r}}_{\text{counter-clockwise along } C} = \underbrace{\int_{\mathrm{A}}^{\mathrm{B}} \vec{F} \cdot d\vec{r}}_{\text{path } \mathrm{C}_1} - \underbrace{\int_{\mathrm{A}}^{\mathrm{B}} \vec{F} \cdot d\vec{r}}_{\text{path } \mathrm{C}_2} = 0$$

[1] An *open surface* is one with an edge, such as a piece of paper or half a tennis ball. Conversely, a *closed surface* is one with no edges, such as an entire tennis ball.

$$\Rightarrow \quad \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r},$$

and the value of the integral is independent of the path chosen (C_1 or C_2). Thus, all the integral can depend upon are the end points, A and B, from which we conclude there is some function, let's call it $U(\vec{r})$, such that:

$$\int_A^B \vec{F} \cdot d\vec{r} \;=\; \underbrace{U(\vec{r}_A) - U(\vec{r}_B)}_{\text{function of end points only}} \;=\; -\int_A^B dU \;=\; -\int_A^B \nabla U \cdot d\vec{r},$$

using Eq. (3.1.2) from the lecture notes ($d\phi = \nabla \cdot d\vec{r}$). Thus,

$$\int_A^B \big(\vec{F} + \nabla U\big) \cdot d\vec{r} \;=\; 0,$$

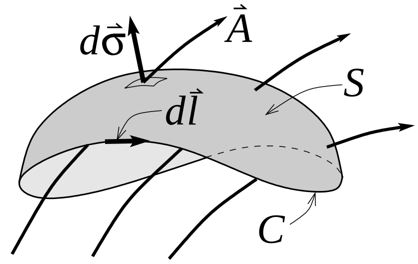
true for any two points, A and B. The only way this integral can be zero regardless of end points chosen is for the integrand to be zero. That is,

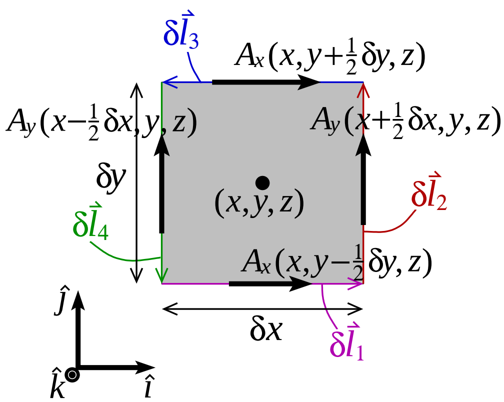
$$\boxed{\vec{F} \;=\; -\nabla U.} \qquad \square$$

An *irrotational* force (where $\nabla \times \vec{F} = 0$) is known as a *conservative force* because a force has to be irrotational for a potential energy function—and thus mechanical energy—to make sense. Only then can we speak of the conservation of mechanical energy.

Stoke's Theorem: *If S is an open surface with perimeter C within a vector field $\vec{A}$, then,*

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S \nabla \times \vec{A} \cdot d\vec{\sigma}. \qquad (1)$$





Proof: Consider an arbitrarily small, flat square centred at (x, y, z) with surface area $\delta\vec{\sigma} = \delta x\, \delta y\, \hat{k}$ embedded in a vector field $\vec{A}$.

Traversing the square in the counter-clockwise direction, its edges and their locations are given by:

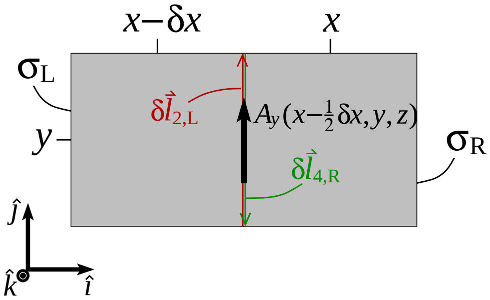
$$\delta\vec{l}_1 = \delta x\, \hat{\imath}, \quad (x, y - \tfrac{1}{2}\delta y, z); \qquad \delta\vec{l}_2 = \delta y\, \hat{\jmath}, \quad (x + \tfrac{1}{2}\delta x, y, z);$$
$$\delta\vec{l}_3 = -\delta x\, \hat{\imath}, \quad (x, y + \tfrac{1}{2}\delta y, z); \qquad \delta\vec{l}_4 = -\delta y\, \hat{\jmath}, \quad (x - \tfrac{1}{2}\delta x, y, z).$$

Summing the quantity $\vec{A} \cdot d\vec{l}$ around the square's perimeter, we get:

$$\begin{aligned}\sum_{n=1}^{4} \vec{A} \cdot \delta\vec{l}_n \;&=\; A_x(x, y - \tfrac{1}{2}\delta y, z)\delta x + A_y(x + \tfrac{1}{2}\delta x, y, z)\delta y \\ &\qquad - A_x(x, y + \tfrac{1}{2}\delta y, z)\delta x - A_y(x - \tfrac{1}{2}\delta x, y, z)\delta y \\ &= \left[\frac{A_y(x + \frac{1}{2}\delta x, y, z) - A_y(x - \frac{1}{2}\delta x, y, z)}{\delta x} \right. \\ &\qquad \left. - \frac{A_x(x, y + \frac{1}{2}\delta y, z) - A_x(x, y - \frac{1}{2}\delta y, z)}{\delta y}\right]\delta x\, \delta y \\ &= \big(\underbrace{\partial_x A_y - \partial_y A_x}_{(\nabla \times \vec{A})_z}\big) d\sigma \;=\; \nabla \times \vec{A} \cdot d\vec{\sigma}, \qquad (2)\end{aligned}$$

in the limit when $\delta x \to dx$, *etc.*

Introduce a second square, $\sigma_{\rm L}$ (left) at $(x-\delta x, y, z)$, that shares the $(x-\frac{1}{2}\delta x, y, z)$ edge with the first square, $\sigma_{\rm R}$ (right) at (x, y, z).



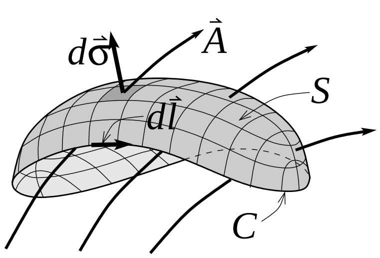
Adding the contributions of the two squares, the RHS of Eq. (2) is the sum of the two values of $(\nabla \times \vec{A}) \cdot \delta\vec{\sigma}$.

Likewise, the LHS of Eq. (2) is the sum of all *eight* edges, although the contributions from the *interior* edges cancel. For example, on edge 2 of $\sigma_{\rm L}$ (red), $\delta\vec{l}_{2,L} \propto +\hat{\jmath}$ while on edge 4 of $\sigma_{\rm R}$ (green), $\delta\vec{l}_{4,R} \propto -\hat{\jmath}$. Thus,

$$\vec{A} \cdot \delta\vec{l}_{2,L} \;=\; -\vec{A} \cdot \delta\vec{l}_{4,R},$$

and the LHS of Eq. (2) is just the sum from all *exterior* edges.

This is true no matter how many squares we bring together, and whether the squares are all actually coplanar *squares*, or non-coplanar "patches"; only the exterior edges (the circumference of the aggregate) of the patches contribute to the LHS of Eq. (2) while *all* patches contribute fully to the RHS.



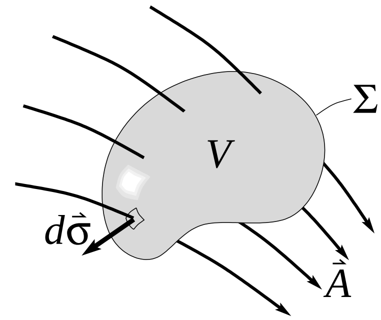
Now, redraw the original figure divided into "Riemann patches" whose aggregate approximates the form of the surface S. Then, Eq. (2) $\Rightarrow$

$$\sum_{\text{exterior edges}} \vec{A} \cdot \delta\vec{l} = \sum_{\text{all patches}} \nabla \times \vec{A} \cdot d\vec{\sigma}, \qquad (3)$$

In the limit as $\delta x \to dx$, the sum on the LHS of Eq. (3) becomes a contour integral and the sum on the RHS becomes an open surface integral:

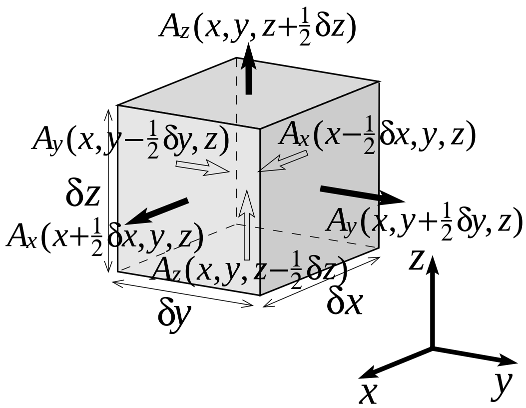
$$\boxed{\oint_C \vec{A} \cdot d\vec{l} = \int_S \nabla \times \vec{A} \cdot d\vec{\sigma}.} \qquad \square$$

And so while we're at it, let's introduce and prove the second of two very important theorems in vector calculus ...



Gauss' Theorem: *If V is a volume with surface area Σ embedded in a vector field $\vec{A}$, then,*

$$\oint_\Sigma \vec{A} \cdot d\vec{\sigma} \;=\; \int_V \nabla \cdot \vec{A}\, dV. \tag{4}$$



Proof: Consider an arbitrarily small cube centred at (x, y, z) of volume $\delta V = \delta x\, \delta y\, \delta z$ embedded within a vector field $\vec{A}$.

The $\hat{\imath}$ face at $(x + \frac{1}{2}\delta x, y, z)$ has area $\delta\vec{\sigma}_x = \delta y\, \delta z\, \hat{\imath}$ (points outside cube). Thus,

$$\vec{A} \cdot \delta\vec{\sigma}_x\Big|_{x+\frac{\delta x}{2}} \;=\; A_x\big(x + \tfrac{1}{2}\delta x, y, z\big)\, \delta y\, \delta z.$$

Similarly, the $\hat{\imath}$ face at $(x - \frac{1}{2}\delta x, y, z)$ has area $\delta\vec{\sigma}_x = \delta y\, \delta z(-\hat{\imath})$ (points outside cube) and,

$$\vec{A} \cdot \delta\vec{\sigma}_x\Big|_{x-\frac{\delta x}{2}} \;=\; -A_x\big(x - \tfrac{1}{2}\delta x, y, z\big)\, \delta y\, \delta z.$$

Thus, their sum is:

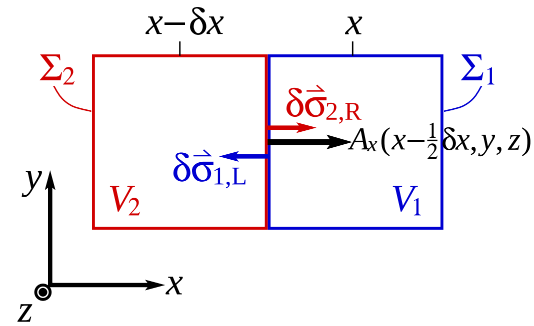
$$\begin{aligned}\vec{A} \cdot \delta\vec{\sigma}_x\Big|_{x+\frac{\delta x}{2}} + \vec{A} \cdot \delta\vec{\sigma}_x\Big|_{x-\frac{\delta x}{2}} \;&=\; \Big(A_x\big(x + \tfrac{1}{2}\delta x, y, z\big) - A_x\big(x - \tfrac{1}{2}\delta x, y, z\big)\Big)\delta y\, \delta z \\ &=\; \frac{\delta A_x}{\delta x}\, \delta x\, \delta y\, \delta z \;=\; \frac{\delta A_x}{\delta x}\, \delta V.\end{aligned}$$

By including the sums on the other four faces, the sum of $\vec{A} \cdot \delta\vec{\sigma}$ over the entire (closed) surface of the cube is:

$$\sum_{\text{cube}} \vec{A} \cdot \delta\vec{\sigma} \;=\; \left(\frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z}\right)\delta V \;=\; \nabla \cdot \vec{A}\, \delta V, \tag{5}$$

in the limit $\delta x \to dx$, *etc.*

Suppose a second cube, Σ_2 (red), at $(x - \delta x, y, z)$ shares the $(x - \frac{1}{2}\delta x, y, z)$ face with the first cube, Σ_1 (blue).



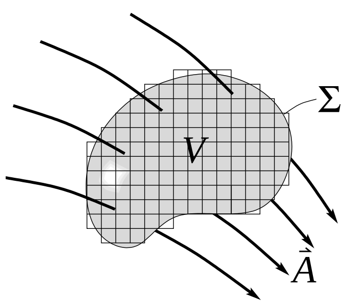
Adding the contributions of the two cubes, the RHS of Eq. (5) is the sum of the two values of $\nabla \cdot \vec{A}\,\delta V$.

Likewise, the LHS of Eq. (5) is the sum of all *twelve* faces, although the contributions from the *interior* faces cancel. For example, the right face of Σ_2, $\delta\vec{\sigma}_{2,R} \propto +\hat{\imath}$ while the left face of Σ_1, $\delta\vec{\sigma}_{1,L} \propto -\hat{\imath}$. Thus,

$$\vec{A} \cdot \delta\vec{\sigma}_{2,R} = -\vec{A} \cdot \delta\vec{\sigma}_{1,L},$$

and the LHS of Eq. (5) is just the sum of all *exterior* faces.

This is true no matter how many cubes we bring together; only the exterior faces (the surface of the aggregate) contribute to the LHS of Eq. (5) while *all* cubes contribute fully to the RHS.



Now, redraw the original figure divided into "Riemann cubes" that approximate the form of the volume V. Writing Eq. (5) for this case, we get:

$$\sum_{\text{exterior faces}} \vec{A} \cdot \delta\vec{\sigma} = \sum_{\text{all cubes}} \nabla \cdot \vec{A}\,\delta V, \tag{6}$$

In the limit as $\delta x \to dx$, the sum on the LHS of Eq. (6) becomes a surface integral and the sum on the RHS becomes a volume integral, leaving us with:

$$\boxed{\oint_{\Sigma} \vec{A} \cdot d\vec{\sigma} = \int_V \nabla \cdot \vec{A}\, dV.} \qquad \square$$

What do Stoke's and Gauss' theorems actually do?

- In univariate calculus, $\int \frac{d}{dx} f(x)dx = f(x) \Rightarrow \frac{d}{dx}$ and $\int$ "annihilate". Stoke's and Gauss' theorems are the multi-variate analogues to this.

- Stoke's theorem,
$$\int_S \nabla \times \vec{A} \cdot d\vec{\sigma} = \oint_C \vec{A} \cdot d\vec{l},$$
turns a surface (double) integral with a differential operator ($\nabla\times$) into a line (single) integral, and one can think of the $\nabla\times$ as "annihilating" one integral.

- Gauss' theorem,
$$\int_V \nabla \cdot \vec{A}\, dV = \oint_\Sigma \vec{A} \cdot d\vec{\sigma},$$
turns a volume (triple) integral with a differential operator ($\nabla \cdot$) into a a surface (double) integral, and one can think of the $\nabla \cdot$ as "annihilating" one integral.