DAMPED HARMONIC MOTION

PHYS 2302, Saint Mary's University D. A. Clarke, July 2019

In class, we found the equation of motion for a damped harmonic oscillator:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x \;=\; 0, \tag{1}$$

which, for initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ has solution:

$$x(t) \;=\; x_0 \begin{cases} e^{-\gamma t}\left(\cosh(qt) + \dfrac{\gamma}{q}\sinh(qt)\right), & \gamma > \omega_0 \text{ (overdamped);} \\ e^{-\gamma t}(1+\gamma t), & \gamma = \omega_0 \text{ (critically damped);} \\ e^{-\gamma t}\,\dfrac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \theta_0), & \gamma < \omega_0 \text{ (underdamped);} \\ \cos(\omega_0 t), & \gamma = 0 \text{ (undamped),} \end{cases} \tag{2}$$

where:

$x_0 \;=\;$ initial displacement of m (stretch of spring);

$\gamma \;=\;$ damping coefficient;

$\omega_0 \;=\; \sqrt{\dfrac{k}{m}} \;=\;$ oscillation frequency of undamped system;

$q \;=\; \sqrt{\gamma^2 - \omega_0^2}$;

$\omega_\mathrm{d} \;=\; iq \;=\; \sqrt{\omega_0^2 - \gamma^2} \;=\;$ oscillation frequency of underdamped system;

$\theta_0 \;=\; \sin^{-1}\dfrac{\gamma}{\omega_0} \;=\;$ phase lag of underdamped system.

To plot Eq. (2), it's easier if we first *scale* it to make all variables *unitless*.

Thus, let $\xi = x/x_0$, $s = \omega_0 t$, and $\alpha = \gamma/\omega_0 \;=\; \sin\theta_0$ (all unitless). Then,

$$\gamma t \;=\; \frac{\gamma}{\omega_0}\,\omega_0 t \;=\; \alpha s; \qquad q \;=\; \omega_0\sqrt{\alpha^2 - 1}; \qquad \omega_\mathrm{d} \;=\; \omega_0\sqrt{1-\alpha^2},$$

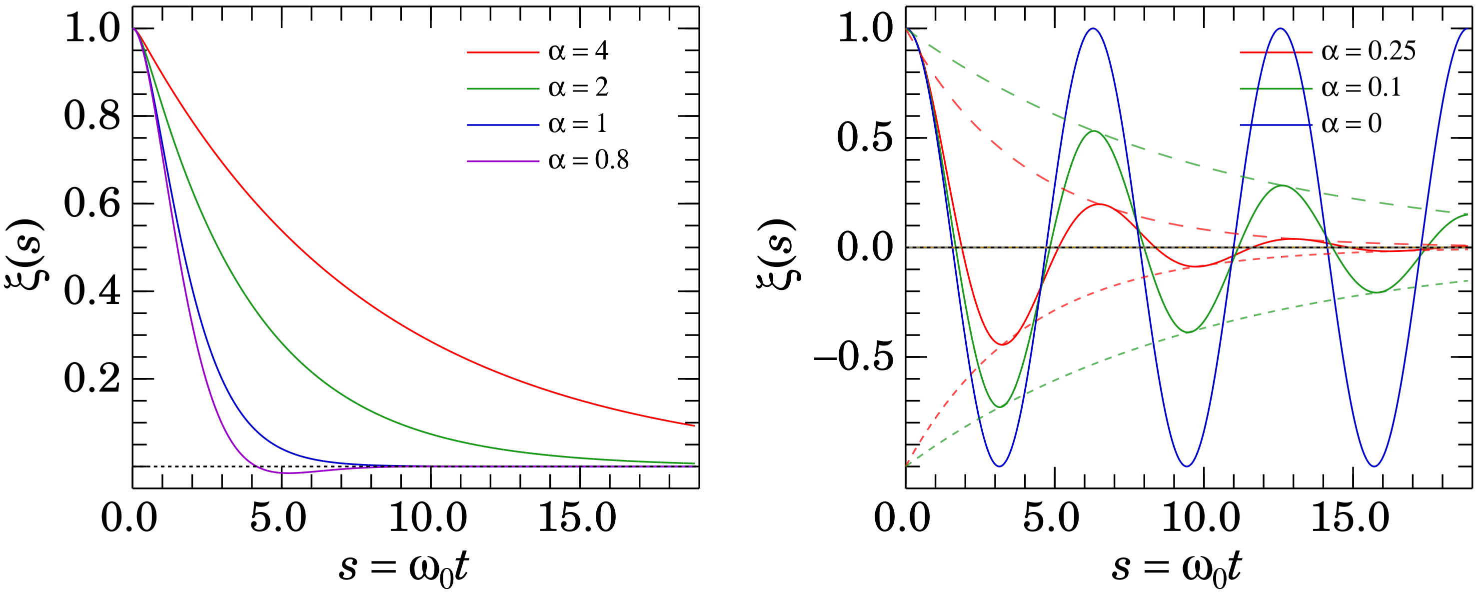
and Eq. (2) becomes:

$$\xi(s) = \begin{cases} e^{-\alpha s}\left(\cosh\left(\sqrt{\alpha^2-1}\,s\right) + \dfrac{\alpha}{\sqrt{\alpha^2-1}}\sinh\left(\sqrt{\alpha^2-1}\,s\right)\right), & \alpha > 1; \\ e^{-s}(1+s), & \alpha = 1; \\ e^{-\alpha s}\,\dfrac{\cos(\sqrt{1-\alpha^2}\,s - \theta_0)}{\sqrt{1-\alpha^2}}, & \alpha < 1; \\ \cos s, & \alpha = 0, \end{cases}$$

which can be plotted by specifying:

- a *domain* for s [*e.g.*, $s = \omega_0 t \in (0, 6\pi \sim 19)$ for three periods];
- α, unitless parameter that determines damping strength:

$$\alpha = \frac{\gamma}{\omega_0}\begin{cases} > 1, & \text{overdamped;} \\ = 1, & \text{critically damped;} \\ < 1, & \text{underdamped;} \\ = 0, & \text{undamped.} \end{cases}$$



Left: overdamped (red and green), critically damped (blue), and sub-critically damped (purple) harmonic oscillators.

Right: under-damped (red and green with dashed envelope functions) and undamped (blue) harmonic oscillators.

Notes.

1. By scaling Eq. (2), we never had to specify m, k, ω_0, x_0, *etc.*
2. For $\alpha > 1$, system reaches equilibrium more slowly than for $\alpha = 1$.
3. Analogue meters (needle and dial) are designed with critical damping to reach equilibrium (the "reading") fastest without oscillating.
4. For $\alpha < 1$ even slightly, oscillator attains some oscillatory behaviour, and "overshoots" equilibrium point.
5. As $\alpha \to 0$, system exhibits increasing oscillatory behaviour with oscillation amplitude given by "envelope function" ($e^{-\alpha s}$).
6. The "phase lag", $\theta_0 = \sin^{-1}\alpha$, means a damped oscillator does not start at a "peak" at $t = 0$; first peak "occurs" at $t = -\theta_0/\omega_0$.