Dimensional Analysis

PHYS 2302, Saint Mary's University D. A. Clarke, July 2019

For three important reasons, physicists are always aware of "dimensions" or "units" as they do calculations, whether they be mks, cgs, or whatever.

1. *As a check on algebra...*

From Eq. (2.6.7) in the lecture notes, the amplitude of a forced, underdamped oscillator is:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}, \tag{1}$$

where: $[\omega_0^2] \equiv$ units of $\omega_0^2 = \mathrm{s}^{-2}$; $[\omega^2] = \mathrm{s}^{-2}$.

Thus, $\left[(\omega_0^2 - \omega^2)^2\right] = \mathrm{s}^{-4} \Rightarrow \left[\gamma^2\omega^2\right] = \mathrm{s}^{-4}$ (can't add apples and oranges).

$\Rightarrow$ $[\gamma^2]$ must be s^{-2} too. So, let's check:

$$\gamma = \frac{b}{2m} \quad \text{where} \quad m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t$$

$$\Rightarrow \quad [b\dot{x}] = \mathrm{N} \text{ (same as } F_0\text{)} = \mathrm{kg\,m\,s^{-2}}$$

$$\Rightarrow \quad [b] = \mathrm{kg\ s^{-1}} \text{ (dividing out } [\dot{x}] = \mathrm{m\,s^{-1}}\text{)}$$

$$\Rightarrow \quad [\gamma] = [b/m] = \mathrm{s}^{-1} \quad \Rightarrow \quad \left[\gamma^2\right] = \mathrm{s}^{-2}, \text{ as desired.}$$

Thus,

$$[\text{RHS of (1)}] = \frac{\mathrm{N/kg}}{\sqrt{\mathrm{s}^{-4}}} = \frac{\cancel{\mathrm{kg}}\,\mathrm{m}\,\cancel{\mathrm{s}^{-2}}}{\cancel{\mathrm{kg}}\,\cancel{\mathrm{s}^{-2}}} = \mathrm{m} = [A] \checkmark$$

Limitations: Checking units won't tell you if your algebra is correct, but it *will* tell you if you've made an error and can point to where!

2. *Arguments of special functions* ($\sin x$, e^x, $\ln x$, *etc.*) must *be unitless!*

From Eq. (2.6.10) in the lecture notes, the time-dependent position of an underdamped, forced oscillator is,

$$x(t) \;=\; -Ae^{-\gamma t}\frac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d}t-\theta)+A\cos(\omega t-\phi),$$

where ω_d is the natural oscillation frequency of the underdamped oscillator; θ and ϕ are phases.

Since $[\gamma]=\mathrm{s}^{-1}$, $[\gamma t]$ (units of exponential argument) $=\mathrm{s}^{-1}\mathrm{s}=1$ (unitless).

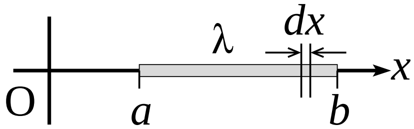
Further, $[\omega_\mathrm{d}t]=[\omega t]$ (units of cosine arguments) $=\mathrm{s}^{-1}\mathrm{s}=1$, also unitless.

Finally, $[\theta]=[\phi]=$ radians, a ratio of two distances and thus unitless.

$\Rightarrow$ arguments of all special functions are unitless.

Occasionally, the math seems to force units on a special function argument.

Example: A line charge with uniform linear charge density, λ, is set up between $x=a$ and b. Find the potential at the origin, O.



Charge in dx: $dq=\lambda dx$.

Potential at O from dq: $dV \;=\; \dfrac{kdq}{x} \;=\; k\lambda\dfrac{dx}{x}$.

$$\Rightarrow\quad V(\mathrm{O}) \;=\; \int_a^b dV \;=\; k\lambda\int_a^b\frac{dx}{x} \;=\; k\lambda\ln x\Big|_a^b \;=\; k\lambda(\ln b-\ln a).$$

But $[a]=[b]=\mathrm{m}$, so does this mean ln is an exception to this rule?

Not really. In this case,

$$\ln b-\ln a \;=\; \ln\left(\frac{b}{a}\right),$$

and $[b/a] = 1$ (unitless)

In physics, if the argument of a special function acquires units, then either:

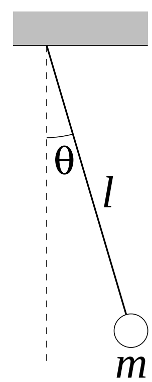
- a simple arithmetical step (such as above) will eliminate the units; or
- an algebraic error has occurred.

In practise, if $[x] = \mathrm{m}$, $[k] = \mathrm{m}^{-1}$, $[t] = \mathrm{s}$, and $[\omega] = \mathrm{s}^{-1}$, then…

$$\ldots\text{we'll always see } \begin{Bmatrix} \sin kx \\ e^{\omega t} \end{Bmatrix}, \text{ and never } \begin{Bmatrix} \sin x \text{ or } \sin k \\ e^{t} \text{ or } e^{\omega} \end{Bmatrix}.$$

3. *Determining simple relationships among quantities*

Example: From Eq. (2.2.7) in the lecture notes, the period of oscillation for a simple pendulum is:



$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Simple relationships such as these can often be foretold just from the units.

a) List all quantities that could possibly influence T:

l, length of cord;

g, acceleration of gravity; identical pendulums have different periods on moon as on earth;

m, shouldn't matter since acceleration in a gravitational field is independent of mass.

That's gotta be it!

b) Set up a power-law relationship between T and influencing variables:

$$T \propto l^\alpha g^\beta,$$

and perform a dimensional analysis:

$$[T] = s^1 = \left[l^\alpha g^\beta\right] = \mathrm{m}^\alpha \mathrm{m}^\beta \mathrm{s}^{-2\beta} = \mathrm{m}^{\alpha+\beta}\mathrm{s}^{-2\beta}$$

$$\Rightarrow \quad \alpha + \beta = 0; \qquad -2\beta = 1.$$

c) Solve for α and β.

$$\beta = -\frac{1}{2}; \quad \alpha = -\beta = -\frac{1}{2} \qquad \Rightarrow \qquad T \propto l^{1/2} g^{-1/2} = \sqrt{\frac{l}{g}},$$

with no physics!

Limitations:

- cannot be used to determine constant of proportionality (in this case, 2π), that's where the physics comes in;
- not easily adapted for multi-term relationships;
- number of variables = number of independent dimensions used by the variables (≤ 4: m, kg, s, A).