Power-series expansions

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In math and physics, we often need to express an expression or part of an expression as a *power series*,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{1}$$
$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 + \cdots .$$

This handout derives and gives you practise using three of the most common types of power series expansions, namely *Taylor*, *Maclaurin*, and *binomial*.

To express a function in the form of Eq. (1) (said to be an *expansion of* $f(x)$ *about the point* x_0), we suppose first that each of $f(x_0)$, $f'(x_0)$, $f''(x_0)$, *etc.*, are known. That is, we know all there is to know about $f(x)$ at x_0.

Then to start, we evaluate Eq. (1) at $x = x_0$ and find: $f(x_0) = a_0$.

Next, taking derivatives of Eq. (1), we find:

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \cdots$$
$$\Rightarrow \quad f'(x_0) = a_1;$$
$$f''(x) = 2a_2 + 6a_3(x - x_0) + 12a_4(x - x_0)^2 + \cdots$$
$$\Rightarrow \quad f''(x_0) = 2a_2, \quad a_2 = \frac{f''(x_0)}{2};$$
$$f'''(x) = 6a_3 + 24a_4(x - x_0) + \cdots$$
$$\Rightarrow \quad f'''(x_0) = 6a_3, \quad a_3 = \frac{f'''(x_0)}{3!},$$

etc. Thus, $a_n = \dfrac{1}{n!} f^{(n)}(x_0)$, and Eq. (1) becomes:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n.} \tag{2}$$

This is the *Taylor expansion* (series) about the point x_0.

The *Maclaurin expansion* is a Taylor expansion about $x_0 = 0$:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n.} \tag{3}$$

Now, consider the Maclaurin expansion of the particular function:

$$f(x) = (1+x)^a \quad \Rightarrow \quad f(0) = 1.$$

Then,

$$\begin{aligned}
f'(x) &= a(1+x)^{a-1} & \Rightarrow \quad f'(0) &= a; \\
f''(x) &= a(a-1)(1+x)^{a-2} & \Rightarrow \quad f''(0) &= a(a-1); \\
f'''(x) &= a(a-1)(a-2)(1+x)^{a-3} & \Rightarrow \quad f'''(0) &= a(a-1)(a-2),
\end{aligned}$$

etc. Thus, Eq. (3) $\Rightarrow$

$$\boxed{(1+x)^a = 1 + ax + \frac{a(a-1)}{2} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \cdots .} \tag{4}$$

This is the *binomial expansion* about $x = 0$, which converges for $|x| < 1$.

Example: Express the relativistic energy, $E = mc^2 = \dfrac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$, as a power series in v^2.

$$\begin{aligned}
E &= m_0 c^2 \left(1 - v^2/c^2\right)^{-1/2} \\
&= m_0 c^2 \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{v^2}{c^2}\right)^2 + \cdots\right]
\end{aligned}$$

$$= m_0c^2\left(1+\frac{v^2}{2c^2}+\frac{3v^4}{8c^4}+\cdots\right) = \underbrace{m_0c^2}_{\substack{\text{rest mass}\\ \text{energy}}}+\underbrace{\frac{1}{2}mv^2}_{\substack{\text{kinetic}\\ \text{energy}}}+\underbrace{\frac{3}{8}m_0\frac{v^4}{c^2}+\cdots}_{\substack{\text{higher-order}\\ \text{relativistic terms}}}.$$

Commonly occurring expansions:

$$\frac{1}{1-x} = (1-x)^{-1} = 1+x+x^2+x^3+\cdots;$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1-x+x^2-x^3+\cdots;$$

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = 1+\frac{x}{2}+\frac{3x^2}{8}+\frac{5x^3}{16}+\cdots.$$

Practise problems:

1. Find a power series expansion (first three terms) for $f(x) = e^x$ about:

   a) $x_0 = 0$; b) $x_0 = 1$.

2. Find a power series expansion for $f(x) = \sin x$ about:

   a) $x_0 = 0$; b) $x_0 = \frac{\pi}{2}$.

3. Find a power series expansion for $f(x) = \cos x$ about $x_0 = 0$.

4. Find a power series expansion for $f(x) = 1 + 2x + x^2$ about:

   a) $x_0 = 0$; b) $x_0 = 1$.

5. Express $f(x) = (x - x_0)^a$ as a power series in:

   a) $\frac{x_0}{x}$; b) $\frac{x}{x_0}$.

For what values of x does each power series converge?

Solutions:

1. a) About $x_0 = 0$, this is a Maclaurin expansion (Eq. 3):

$$f(x) = e^x \Rightarrow f(0) = 1;$$

$$f'(x) = e^x \Rightarrow f'(0) = 1;$$

$$f''(x) = e^x \Rightarrow f''(0) = 1, \ \textit{etc.},$$

$$\Rightarrow \boxed{e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots .} \tag{5}$$

b) About $x_0 = 1$, this is a Taylor expansion (Eq. 2):

$$f(1) = f'(1) = f''(1) = \cdots = e$$

$$\Rightarrow \boxed{e^x = e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{3!} + \cdots .}$$

Note that dividing through by e, we'd get:

$$\frac{e^x}{e} = e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \cdots ,$$

which we could have written down directly from Eq. (5).

2. a) About $x_0 = 0$, this is a Maclaurin expansion (Eq. 3):

$$f(x) = \sin x \Rightarrow f(0) = 0;$$

$$f'(x) = \cos x \Rightarrow f'(0) = 1;$$

$$f''(x) = -\sin x \Rightarrow f''(0) = 0;$$

$$f'''(x) = -\cos x \Rightarrow f'''(0) = -1;$$

$$f^{(iv)}(x) = \sin x \Rightarrow f^{(iv)}(0) = 0;$$

$$f^{(v)}(x) = \cos x \Rightarrow f^{(v)}(0) = 1, \ \textit{etc.},$$

$$\Rightarrow \quad \boxed{\sin x \,=\, x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \,.}$$

Note how many derivatives we had to evaluate to get three non-zero terms.

b) About $x_0 = \pi/2$, this is a Taylor series:

$$f(\pi/2) \,=\, -f''(\pi/2) \,=\, f^{(iv)}(\pi/2) \,=\, \cdots \,=\, 1;$$

$$f'(\pi/2) \,=\, -f'''(\pi/2) \,=\, f^{(v)}(\pi/2) \,=\, \cdots \,=\, 0$$

$$\Rightarrow \quad \boxed{\sin x \,=\, 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{4!} + \cdots \,.} \qquad (6)$$

3. About $x_0 = 0$, this is a Maclaurin expansion (Eq. 3):

$$f(x) \,=\, \cos x \quad \Rightarrow \quad f(0) \,=\, 1;$$

$$f'(x) \,=\, -\sin x \quad \Rightarrow \quad f'(0) \,=\, 0;$$

$$f''(x) \,=\, -\cos x \quad \Rightarrow \quad f''(0) \,=\, -1;$$

$$f'''(x) \,=\, \sin x \quad \Rightarrow \quad f'''(0) \,=\, 0;$$

$$f^{(iv)}(x) \,=\, \cos x \quad \Rightarrow \quad f^{(iv)}(0) \,=\, 1, \ \textit{etc.},$$

$$\Rightarrow \quad \boxed{\cos x \,=\, 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \,.} \qquad (7)$$

Note that had we made the substitution $y = x - \pi/2$ in Eq. (6), we would get:

$$\sin(y + \pi/2) \,=\, \cos y \,=\, 1 - \frac{y^2}{2} + \frac{y^4}{4!} - \cdots \,,$$

giving us an alternate way to Eq. (7).

4. a) $f(x) = 1 + 2x + x^2 = 1 + 2(x-0) + (x-0)^2$ is already it's own power series about $x_0 = 0$!

b) About $x = 1$ is a different story. To achieve this, we just "force it":

$$f(x) = 1 + 2x + x^2 = 1 + 2(x-1+1) + (x-1+1)^2$$
$$= 1 + 2(x-1) + 2 + (x-1)^2 + 2(x-1) + 1 = 4 + 4(x-1) + (x-1)^2.$$

5. a) Here, a power series is best obtained from the binomial expansion, Eq. (4). To obtain a power series in x_0/x, factor out x to get:

$$f(x) = x^a\Big(1 - \frac{x_0}{x}\Big)^a = x_0^a\,\frac{x^a}{x_0^a}\left[1 + a\Big(-\frac{x_0}{x}\Big) + \frac{1}{2}a(a-1)\Big(-\frac{x_0}{x}\Big)^2 + \ldots\right]$$
$$= x_0^a\left(\frac{x_0}{x}\right)^{-a}\left[1 - a\frac{x_0}{x} + \frac{1}{2}a(a-1)\left(\frac{x_0}{x}\right)^2 + \ldots\right]$$
$$= x_0^a\left[\left(\frac{x_0}{x}\right)^{-a} - a\left(\frac{x_0}{x}\right)^{-a+1} + \frac{1}{2}a(a-1)\left(\frac{x_0}{x}\right)^{-a+2} + \ldots\right].$$

This power series would be suitable (*i.e.*, would converge) for $x > x_0$.

b) To obtain a power series in x/x_0, factor out $-x_0$ to get:

$$f(x) = (-x_0)^a\Big(1 - \frac{x}{x_0}\Big)^a = (-x_0)^a\left[1 + a\Big(-\frac{x}{x_0}\Big) + \frac{1}{2}a(a-1)\Big(-\frac{x}{x_0}\Big)^2 + \ldots\right]$$
$$= (-x_0)^a\left[1 - a\frac{x}{x_0} + \frac{1}{2}a(a-1)\left(\frac{x}{x_0}\right)^2 + \ldots\right].$$

This power series would be suitable (*i.e.*, would converge) for $x < x_0$.