PHYS 2302 PRACTISE FINAL EXAM

Instructions

For a complete paper, you must attempt both problems from part A, and two of the three problems in part B. Each problem is worth 20 points and, unless otherwise indicated, each part of a problem is of equal weight.

If you attempt all three problems in part B, the solution receiving the lowest grade will count at half weight as bonus points (maximum 10). Maximum score on this exam is 90/80.

This is a closed-book exam; notebooks, textbooks, *etc.*, are not permitted. A simple scientific calculator may be used; all other electronic devices should be turned off and put away.

Formulæ are provided on the last four pages of the exam and may be torn off the exam for ready reference, if convenient. Including the formula sheets, there are eight pages to this exam.

Solutions are to be written in the exam booklet provided. This exam, including the formula sheets, must be turned in with the exam booklet.

You have up to three hours to complete this exam.

Good luck and have a well-deserved holiday!

Part A: do both problems.

Problem 1. Damped, driven harmonic oscillator: A damped harmonic oscillator is set into motion, and after n complete oscillations, its amplitude is half the initial amplitude. It is further observed that when the oscillator is driven by a driving force $F = F_0 \cos \omega t$ for varying ω, its resonance peak (on a plot of amplitude squared, A^2, *vs.* the driving frequency, ω) has a full-width-half-max (FWHM) $\Delta\omega$, as defined on the formula sheets.

In terms of n and/or $\Delta\omega$:

a) find the quality factor, $Q_{\rm d}$;

b) without assuming $\gamma \ll \omega_0$, find the damping coefficient, γ;

c) find the natural oscillation period of the damped harmonic oscillator, $\omega_{\rm d}$; and

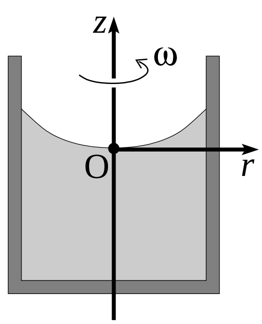
d) show that the resonant frequency, $\omega_{\rm r}$, is given by:

$$\omega_{\rm r}^2 = \frac{\Delta\omega}{4}\left(\frac{2\pi n}{\ln 2} - \frac{\ln 2}{2\pi n}\right).$$

e) If the damped oscillator is a mass m on a spring with spring constant k, what is k in terms of n, $\Delta\omega$, and m?

Recall for a Hooke's spring, the undamped frequency of oscillation is given by $\omega_0^2 = \dfrac{k}{m}$.

Problem 2. Mach bucket: A bucket of water spins at an angular speed ω about its symmetry axis. As shown in the figure, an inertial coordinate system, O, is defined such that z is aligned with the symmetry axis and r is the radial distance from the axis. You are to find the function, $z(r)$, that describes the spinning surface of the water in two ways.



a) (12 points) Assess all real and inertial forces acting on a droplet of water of mass m, say, at rest on the rotating surface using a rotating coordinate system, O′, whose origin and z'-axis coincide with the origin and z-axis of O, and whose r' axis rotates with the water so that m remains at rest in the r'-z' plane.

Hint: If θ is the tangent angle to the water surface at m, $\tan\theta = dz'/dr' = dz/dr$.

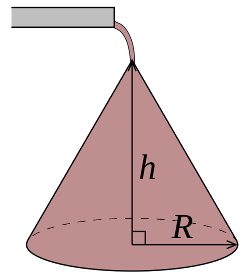
b) (8 points) Assess only the real forces acting on m from the inertial coordinate system, O, and once again find $z(r)$.

Obviously, the two methods should give identical results.

Part B: choose any two problems; the third counts for bonus points.

Problem 3. A cone-shaped sandpile: A conveyor belt piles sand into a "right cone" of height h and base radius R, as depicted in the figure.

If the coefficient of static friction between each layer of sand along the slope and the sand beneath it (along which the sand might slip) is μ_s, what is the maximum volume of sand in the cone for a given R if none of the sand slips? Your final answer should be in terms of μ_s and R.



Hint: The volume of a right cone is $\dfrac{\pi}{3}R^2h$.

Problem 4. Air drag on a bullet: A gun is fired vertically upward, and the bullet experiences an air drag given by $f_d = bv^2$, where v is the speed of the bullet, b is a constant, and where f_d is directed opposite to the direction of motion. Let z be the vertical coordinate.

a) (7 points) Ignoring the effects of earth's rotation, show that the function $v_{up}(z)$ of the bullet on its way up is given by:

$$v_{up}^2(z) = Ae^{-2kz} - \frac{g}{k},$$

where $k = b/m$ and A is an arbitrary constant.

Hint: From an FBD, show that Newton's second law can be written as the separable first order ODE, $\frac{1}{2}d\phi/dz = -k\phi - g$, where $\phi = v^2$. Then, if you've forgotten how to solve a separable ODE, see the reminder on the formula sheets.

b) (5 points) Show that the function $v_{down}(z)$ of the bullet on its way down is given by:

$$v_{down}^2(z) = Be^{2kz} + \frac{g}{k},$$

where B is another arbitrary constant.

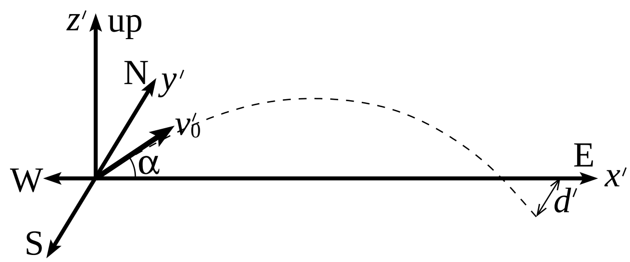
c) (5 points) If the muzzle speed of the bullet is v_0, find A and B.

Hint: To find A, set $v_{up}(0) = v_0$. To find B, note that $v_{up}(h) = v_{down}(h) = 0$.

d) (3 points) What is the speed of the bullet when it returns to the height of the muzzle on its way down? (*Hint*: It's not v_0!)

Problem 5. Inertial deflection of a bullet: A bullet is fired from a rifle with a muzzle speed v_0' due east at latitude λ and an elevation angle α (angle between the rifle's long axis and the horizontal).

a) (15 points) Ignoring air resistance and how gravity varies with height, show that the bullet hits the earth with a lateral deflection, d', given by:



$$d' = -\sqrt{\frac{2R_0^3 \tan\alpha}{g}}\,\omega\sin\lambda,$$

dropping all terms proportional to ω^2 or higher power. Here, the negative sign means the drift is southward, $R_0 = (v_0'^2 \sin 2\alpha)/g$ is the rifle's range not accounting for the earth's rotation, and ω is the rotation speed of the earth.

b) (5 points) For $v_0' = 500\,\mathrm{m\,s^{-1}}$, $\alpha = 30°$, and $\lambda = 45°$, find a numerical value for d'/R_0, the deflection as a fraction of its approximate range. Take the earth's rotation speed to be $7.292 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$ and $g = 9.81\,\mathrm{m\,s^{-2}}$.

FORMULA SHEETS

1. Variations of **Newton's Second Law** for rectilinear (1-D) motion:

$$\sum F = ma = m\ddot{x} = m\dot{v} = m\frac{dv}{dx}v = \frac{m}{2}\frac{dv^2}{dx}.$$

2. **Frictional forces** (Coulomb model):

$$f_\mathrm{k} = \mu_\mathrm{k} N; \qquad f_\mathrm{s} \le \mu_\mathrm{s} N; \qquad D = c_1 v + c_2 v^2,$$

where f_k, f_s, and D are the kinetic friction force, static friction force, and "air drag" respectively, N is the normal force, v is the velocity of a particle through the air, and where the remaining quantities are coefficients.

3. **Centripetal acceleration:** For an object moving at speed v in a circular path of radius r, its centripetal acceleration directed toward the centre of curvature is,

$$a_\mathrm{cp} = \frac{v^2}{r} = \omega^2 r,$$

where $\omega = v/r$ is the angular speed.

4. **Period of oscillation:** If the angular frequency of oscillation is ω ($\mathrm{rad\,s^{-1}}$), the period of oscillation in seconds is:

$$T = \frac{2\pi}{\omega}.$$

5. **Damped harmonic oscillator:** The differential equation of motion for a damped harmonic oscillator is,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \tag{1}$$

where: γ = damping coefficient;
ω_0 = angular frequency of the undamped oscillator.

For initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$, the solution to equation (1) is:

$$x(t) = \begin{cases} x_0 e^{-\gamma t}\left(\cosh qt + \dfrac{\gamma}{q}\sinh qt\right), & \gamma > \omega_0 \text{ (overdamped)}; \\ x_0 e^{-\gamma t}(1+\gamma t), & \gamma = \omega_0 \text{ (critically damped)}; \\ x_0 e^{-\gamma t}\,\dfrac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \theta_0), & \gamma < \omega_0 \text{ (underdamped)}; \\ x_0 \cos\omega_0 t, & \gamma = 0 \text{ (undamped)}, \end{cases}$$

where: x_0 = initial displacement of oscillator;

$q = \sqrt{\gamma^2 - \omega_0^2}$;

$\omega_{\rm d} = \sqrt{\omega_0^2 - \gamma^2}$ = angular frequency of damped oscillator;

$\theta_0 = \sin^{-1} \dfrac{\gamma}{\omega_0}$ = phase lag of damped oscillator.

6. **Quality factor of a damped harmonic oscillator**, $Q_{\rm d}$, is a measure of how slowly the total energy of an oscillator is lost to damping. For underdamped systems,

$$Q_{\rm d} \approx \frac{\omega_{\rm d}}{2\gamma},$$

where: $\omega_{\rm d}$ = angular frequency of the damped oscillator;

$\gamma = \sqrt{\omega_0^2 - \omega_{\rm d}^2}$, the damping coefficient;

ω_0 = angular frequency of the undamped oscillator.

7. **Resonance:** Maximum amplitude of a driven, damped oscillator is: $A_{\rm max} = \dfrac{F_0}{2m\gamma\omega_{\rm d}}$, when driven at the *resonant frequency*,

$$\omega_{\rm r} = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\omega_{\rm d}^2 - \gamma^2},$$

where: F_0 = amplitude of the driving force;

m = mass of the oscillator;

γ = damping coefficient;

$\omega_{\rm d} = \sqrt{\omega_0^2 - \gamma^2}$ = natural angular frequency of damped oscillator;

ω_0 = natural angular frequency of undamped oscillator.

Sharpness of resonance is defined as,

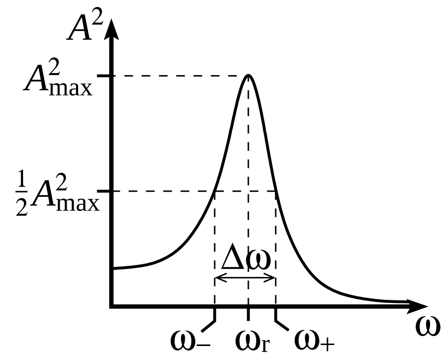
$$S = \frac{\Delta\omega}{\omega_{\rm r}} = \frac{\omega_+ - \omega_-}{\omega_{\rm r}} \approx \frac{1}{Q_{\rm d}} \text{ for } \gamma \ll \omega_0,$$

where: $\Delta\omega$ = full-width-half-maximum (FWHM);

$\omega_\pm^2 = \omega_{\rm r}^2 \pm 2\gamma\omega_{\rm d}$ (note the squares!)

= driving frequencies where $A^2 = \frac{1}{2}A_{\rm max}^2$;

$Q_{\rm d} = \dfrac{\omega_{\rm d}}{2\gamma}$ = quality factor.

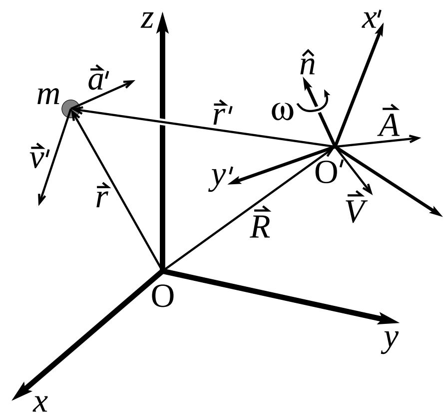


8. **Non-inertial frames of reference and Coriolis' theorem**: Kinematical quantities in an inertial (O) and non-inertial (O′) frames of reference are related by:

$$\vec{r} = \vec{r}\,' + \vec{R};$$

$$\vec{v} = \vec{v}\,' + \vec{\omega} \times \vec{r}\,' + \vec{V};$$

$$\vec{a} = \vec{a}\,' + \dot{\vec{\omega}} \times \vec{r}\,' + 2\,\vec{\omega} \times \vec{v}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') + \vec{A},$$



where: $\vec{r}$ = position of m relative to O;
$\vec{r}\,'$ = position of m relative to O′;
$\vec{R}$ = position of O′ relative to O;
$\vec{v}$ = velocity of m relative to O;
$\vec{v}\,'$ = velocity of m relative to O′;
$\vec{V}$ = velocity of O′ relative to O;
$\vec{\omega}$ = angular velocity of O′ relative to O about a fixed axis (*e.g.*, $\hat{n}$);
$\vec{a}$ = acceleration of m relative to O;
$\vec{a}\,'$ = acceleration of m relative to O′; and
$\vec{A}$ = acceleration of O′ relative to O,

and where the *inertial* accelerations are defined as follows:

$-\dot{\vec{\omega}} \times \vec{r}\,'$ = *transverse* acceleration; $-2\,\vec{\omega} \times \vec{v}\,'$ = *Coriolis* acceleration;
$-\vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$ = *centrifugal* acceleration; $\vec{A}$ = *translational* acceleration.

Coriolis' theorem: $\boxed{\vec{F}\,' = m\vec{a}\,' = \vec{F} - m\,\dot{\vec{\omega}} \times \vec{r}\,' - 2m\,\vec{\omega} \times \vec{v}\,' - m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') - m\vec{A},}$

where:

$\vec{F}\,'$ = forces observed in O′; $\vec{F} = m\vec{a}$, real forces observed in O;
$-m\,\dot{\vec{\omega}} \times \vec{r}\,'$ = *transverse* force, $\vec{F}_\perp$; $-2m\,\vec{\omega} \times \vec{v}\,'$ = *Coriolis* force, $\vec{F}_{\rm Cor}$;
$-m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$ = *centrifugal* force, $\vec{F}_{\rm cent}$; $-m\vec{A}$ = *translational* force, $\vec{F}_{\rm tran}$.

9. **Projectile motion near the surface of the earth** (a rotating frame of reference): In the absence of air resistance, the position components of a projectile are given by:

$$x'(t) = x'_0 + \dot{x}'_0 t - \omega t^2(\dot{z}'_0 \cos\lambda - \dot{y}'_0 \sin\lambda) + \tfrac{1}{3}\omega g t^3 \cos\lambda; \qquad \text{(east-west)}$$

$$y'(t) = y'_0 + \dot{y}'_0 t - \omega \dot{x}'_0 t^2 \sin\lambda; \qquad \text{(north-south)}$$

$$z'(t) = z'_0 + \dot{z}'_0 t - \tfrac{1}{2} g t^2 + \omega \dot{x}'_0 t^2 \cos\lambda, \qquad \text{(up-down)}$$

where (x'_0, y'_0, z'_0) is the initial position of the projectile, $(\dot{x}'_0, \dot{y}'_0, \dot{z}'_0)$ is its initial velocity, ω is the rotation speed of the earth (7.292×10^{-5} rad s^{-1}), and λ is the latitude.

10. Some **trigonometric identities**:

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}; \qquad \sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi;$$

$$\sin 2\theta = 2\sin\theta\cos\theta;$$

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}; \qquad \cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi;$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta;$$

$$\tan\theta = \cot\left(\frac{\pi}{2}-\theta\right); \qquad \tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi};$$

$$\cot\theta = \tan\left(\frac{\pi}{2}-\theta\right); \qquad \cot(\theta+\phi) = \frac{\cot\theta\cot\phi-1}{\cot\theta+\cot\phi}.$$

11. **Solving a first-order ODE by separation of variables**: A first order ordinary differential equation of the form,

$$\frac{dy}{dx} = \frac{f(x)}{g(y)},$$

where $f(x)$ and $g(y)$ are arbitrary functions is *separable*, and can be rewritten as,

$$g(y)\,dy = f(x)\,dx \quad \Rightarrow \quad \int g(y)\,dy = \int f(x)\,dx.$$