Forced harmonic motion

PHYS 2302, Saint Mary's University — D. A. Clarke, July 2019

In class, we found the equation of motion for a driven harmonic oscillator:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x \;=\; \frac{F_0}{m}\cos\omega t, \tag{1}$$

which, for initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ has solution:

$$x(t) \;=\; \underbrace{-Ae^{-\gamma t}\,\frac{\omega_0}{\omega_\mathrm{d}}\,\cos(\omega_\mathrm{d} t - \theta)}_{x_\mathrm{h};\ \text{transient}} + \underbrace{A\cos(\omega t - \phi)}_{x_\mathrm{p};\ \text{steady-state}} \tag{2}$$

where:

$$\gamma \;=\; \text{damping coefficient;}$$

$$\omega_0 \;=\; \text{oscillation frequency of undamped system;}$$

$$\omega_\mathrm{d} \;=\; \sqrt{\omega_0^2 - \gamma^2} \;=\; \text{oscillation frequency of underdamped system;}$$

$$A \;=\; \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2) + 4\gamma^2\omega^2}} \;=\; \text{amplitude of oscillation;}$$

$$\theta \;=\; \tan^{-1}\frac{\gamma(\omega_0^2 + \omega^2)}{\omega_\mathrm{d}(\omega_0^2 - \omega^2)} \;=\; \text{phase lag of transient term, } x_\mathrm{h}(t);$$

$$\phi \;=\; \tan^{-1}\frac{2\gamma\omega}{\omega_0^2 - \omega^2} \;=\; \text{phase lag of steady-state term, } x_\mathrm{p}(t).$$

Here, $\gamma < \omega_0$ (underdamped).

To scale Eq. (2), let $\xi = x/A$, $s = \omega_0 t$, $\alpha = \gamma/\omega_0$, and $\beta = \omega/\omega_0$ (all unitless). Then,

$$\gamma t \;=\; \frac{\gamma}{\omega_0}\,\omega_0 t \;=\; \alpha s; \qquad \omega_\mathrm{d} \;=\; \omega_0\sqrt{1 - \alpha^2}; \qquad \omega t \;=\; \frac{\omega}{\omega_0}\,\omega_0 t \;=\; \beta s,$$

and Eq. (2) becomes:

$$\xi(t) \;=\; -\frac{e^{-\alpha s}}{\sqrt{1-\alpha^2}}\cos\left(\sqrt{1-\alpha^2}\,s-\theta\right)+\cos(\beta s-\phi),$$

where,

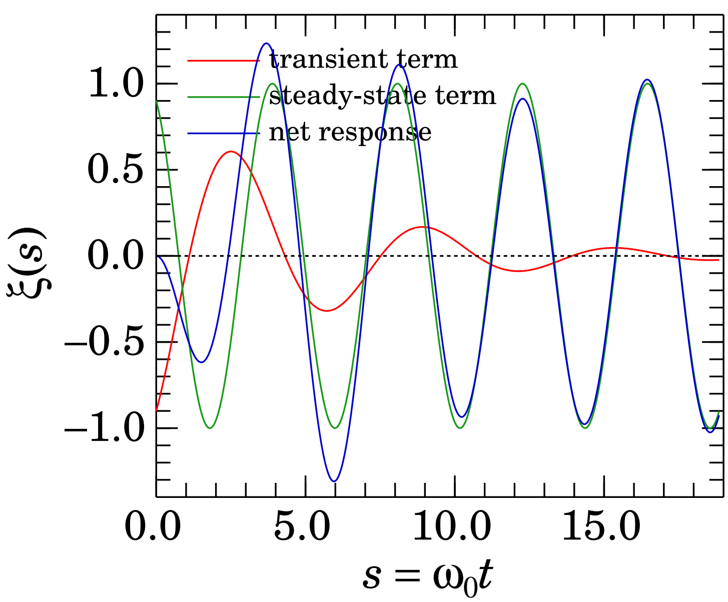
$$\theta \;=\; \tan^{-1}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\,\frac{1+\beta^2}{1-\beta^2}\right); \qquad \phi \;=\; \tan^{-1}\frac{2\alpha\beta}{1-\beta^2}.$$

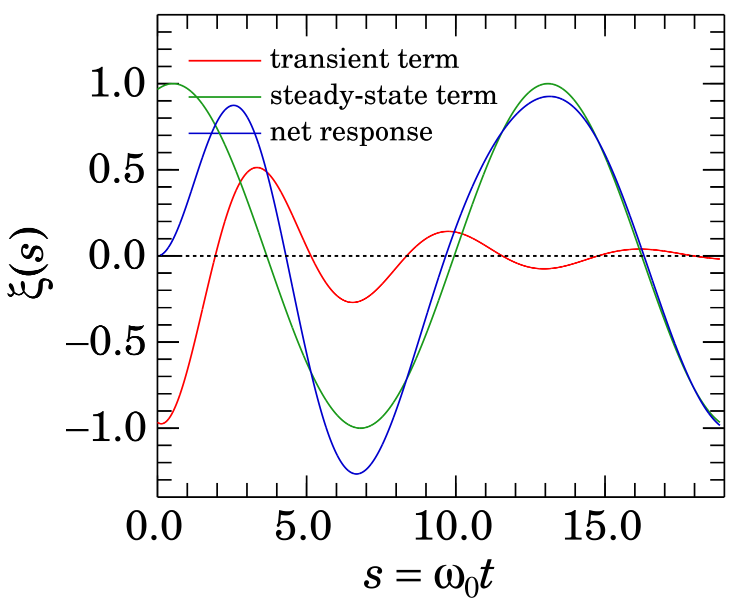
This can be plotted by specifying:

- a *domain* for s [*e.g.*, $s=\omega_0 t \in (0, 6\pi \sim 19)$ for three periods at ω_0];

- α, unitless damping coefficient (0.2 for both plots);

- β, unitless driving frequency (1.5 left, 0.5 right).





Notes.

- For $\beta > 1$ (left), phases $\theta, \phi < 0 \;\Rightarrow\;$ first peak is left of $s = 0$.

- For $\beta < 1$ (right), phases $\theta, \phi > 0 \;\Rightarrow\;$ first peak is right of $s = 0$.