Lessons for

PHYS 2302
Mechanics I

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Preface

This document contains the 22 lessons I created from my lecture notes for PHYS 2302, an introductory sophomore mechanics course based on the text by Fowles and Cassiday *Analytical Mechanics* (7th ed., ISBN 0-534-49492-7). In addition to the material from the lecture notes, each lesson contains a *Lesson page* indicating to the student what the purpose of each lesson is, and all but the last lesson have a "mini-tutorial" at their end. While these were developed in response to the Covid-19 pandemic and the need to put all classes "on-line", I have decided to try this "flipped-classroom" format when we return to in-person classes.

In this first of two courses, I cover Chapters 2–5 with bits from chapter 1 inserted when needed. There are also three segments in these notes based on solving very elementary ODEs. The follow-on course, PHYS 2303 (Mechanics II) carries on with Chapters 6, 7, 8, and 11. Chapters 9 and 10 are largely the subject of PHYS 3300 (Classical Mechanics), and skipped completely.

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Lesson 1

This lesson reviews the fundamental properties of vectors, where we will:

1. make the distinction between a vector and a scalar;
2. learn to express vectors as ordered triples and in terms of unit vectors;
3. review basic vector operations and identities;
4. introduce *vector projections*; and
5. review how to differentiate common vector combinations.

Part I: Introduction and Review

Reading assignment: Chapters 1 and 2

1.1 Vector fundamentals (FC §1.3 – 1.7, 1.9)

A *scalar*, a, is a magnitude; a single number with units to answer questions:

- how much? (\$25);
- how tall? (10 m);
- what's the temperature? (-10° C); *etc.*

A *vector*, $\vec{A}$, has both magnitude and direction. It can be expressed as:

- a single number with units and an explicit direction: $\vec{v} = 100\,\mathrm{km/hr}$ NE;
- as an *ordered triple* (in 3-D): $\vec{F} = (1.2, 1.6, -1.5)\,\mathrm{N}$, three *components* relative to a defined *coordinate system*,

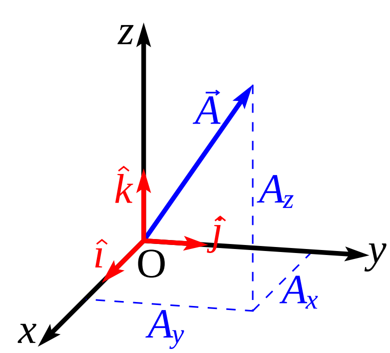
and answers questions like:

- what's your velocity? $\vec{v} = 100\,\mathrm{km/hr}$ NE;
- what force is exerted? $\vec{F} = (1.2, 1.6, -1.5)\,\mathrm{N}$; *i.e.* 1.2 N forward, 1.6 N rightward, 1.5 N downward.

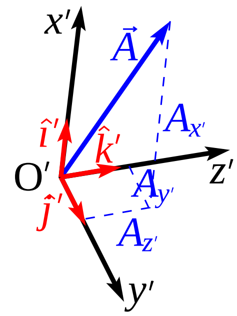
In a 3-D Cartesian coordinate system, O, a vector has three components:

$$\vec{A} = (A_x, A_y, A_z) = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k},$$

where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are *unit* vectors (length = 1) in x-, y-, z-directions.



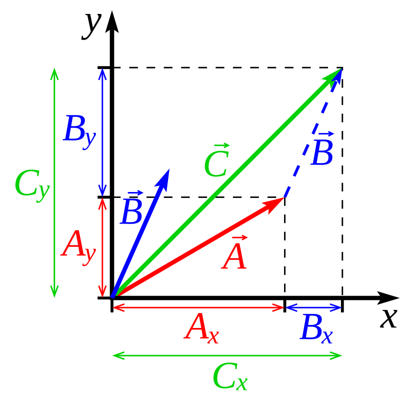
Rotating or translating O $\to$ O$'$ changes components, but not the vector:



$$\vec{A} \;=\; A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \;=\; A_{x'}\hat{\imath}' + A_{y'}\hat{\jmath}' + A_{z'}\hat{k}'.$$

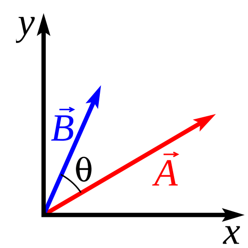
$A_x \neq A_{x'}$, *etc.*, since $\hat{\imath} \neq \hat{\imath}'$, *etc.*, but $\vec{A}$ is the same.

Let a be a scalar and $\vec{A}$, $\vec{B}$, $\vec{C}$ be vectors with components (A_x, A_y, A_z), (B_x, B_y, B_z), (C_x, C_y, C_z) relative to same coordinate system.



1. $\vec{A} = \vec{B} \iff A_x = B_x,\ A_y = B_y,\ A_z = B_z$
2. $\vec{C} = \vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$.
3. $a\vec{A} = (aA_x, aA_y, aA_z)$.
4. Cartesian unit vectors: $\hat{\imath} = \hat{e}_x = \hat{x} = (1, 0, 0)$;

   $\hat{\jmath} = \hat{e}_y = \hat{y} = (0, 1, 0)$; $\quad \hat{k} = \hat{e}_z = \hat{z} = (0, 0, 1)$.
5. scalar (dot, inner) product:



$$\vec{A}\cdot\vec{B} \;=\; \vec{B}\cdot\vec{A} \;=\; A_xB_x + A_yB_y + A_zB_z \;=\; AB\cos\theta$$

$$\Rightarrow \qquad \cos\theta \;=\; \frac{A_xB_x + A_yB_y + A_zB_z}{AB}.$$

$$\hat{\imath}\cdot\hat{\jmath} \;=\; \hat{\jmath}\cdot\hat{k} \;=\; \hat{k}\cdot\hat{\jmath} \;=\; 0; \qquad \hat{\imath}\cdot\hat{\imath} \;=\; \hat{\jmath}\cdot\hat{\jmath} \;=\; \hat{k}\cdot\hat{k} \;=\; 1.$$

6. vector magnitude: $A \;=\; |\vec{A}| \;=\; \sqrt{\vec{A}\cdot\vec{A}} \;=\; \sqrt{A_x^2 + A_y^2 + A_z^2}$.

   For $\vec{F} = (1.2, 1.6, -1.5)\,\mathrm{N}$, $F = \sqrt{1.2^2 + 1.6^2 + (-1.5)^2} = 2.5\,\mathrm{N}$.
7. vector (cross) product:

$$\vec{A}\times\vec{B} \;=\; (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)$$

$$=\; AB\sin\theta \;\; \text{(direction given by \textit{right hand rule})}.$$

$$\vec{A}\times\vec{A} = 0: \quad \hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0.$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}: \quad \hat{\imath}\times\hat{\jmath} = \hat{k}; \quad \hat{\jmath}\times\hat{k} = \hat{\imath}; \quad \hat{k}\times\hat{\imath} = \hat{\jmath};$$

$$\hat{\jmath}\times\hat{\imath} = -\hat{k}; \quad \hat{k}\times\hat{\jmath} = -\hat{\imath}; \quad \hat{\imath}\times\hat{k} = -\hat{\jmath}.$$

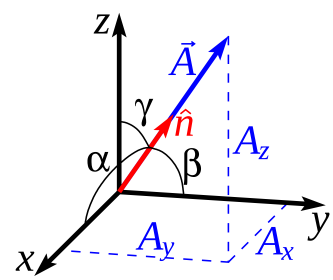
8. triple products:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A}) \quad \text{“cyclic rule”};$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \quad \text{“back-cab rule”}.$$

Exercise: Show that $\vec{A}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{A}\times\vec{B}) = 0$.

9. Unit vector in direction of $\vec{A}$: projections.



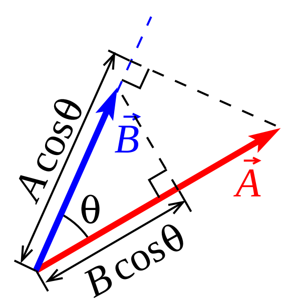
$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} = A\left(\frac{A_x}{A}\hat{\imath} + \frac{A_y}{A}\hat{\jmath} + \frac{A_z}{A}\hat{k}\right)$$

$$= A\big(\underbrace{\cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k}}_{\hat{n}}\big) = A\hat{n},$$

where: $\cos\alpha = A_x/A$, $\cos\beta = A_y/A$, $\cos\gamma = A_z/A$ are *direction cosines*;

$\hat{n}$ is a unit vector $\parallel \vec{A}$. (*Exercise*: Show that $\hat{n}\cdot\hat{n} = 1$.)

Thus,



$$\vec{B}\cdot\vec{A} = BA\cos\theta = \vec{B}\cdot(A\hat{n}) = A\vec{B}\cdot\hat{n}$$

$$\Rightarrow \quad \vec{B}\cdot\hat{n} = B\cos\theta,$$

is *projection* of $\vec{B}$ onto $\vec{A}$.

Similarly, $A\cos\theta$ is projection of $\vec{A}$ onto $\vec{B}$.

10. Derivatives. If $\vec{A}$, $\vec{B}$, a are all functions of t, then:

$$\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt};$$

$$\frac{d}{dt}(a\vec{A}) = \frac{da}{dt}\vec{A} + a\frac{d\vec{A}}{dt};$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B}+\vec{A}\cdot\frac{d\vec{B}}{dt} \quad \text{(order is unimportant);}$$
$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt} \quad \text{(order is important).}$$

Tutorial 1.1

Problem 1 In some Cartesian coordinate system, let $\vec{A}=(3,-12,4)$ and $\vec{B}=4\hat{\imath}+4\hat{\jmath}-7\hat{k}$. Evaluate:

a) $\vec{A}+\vec{B}$ in unit vector notation;

b) $\vec{A}-\vec{B}$ as an ordered triple;

c) $\vec{A}\times\vec{B}$ in unit vector notation;

d) $\vec{A}\cdot\vec{B}$;

e) the angle between $\vec{A}$ and $\vec{B}$; and

f) the projection of $\vec{A}$ onto $\vec{B}$.

Problem 2 (FC 1.10) Show that the area of the parallelogram defined by vectors $\vec{A}$ and $\vec{B}$ is $|\vec{A}\times\vec{B}|$.

Lesson 2

This lesson introduces three important coordinate systems: Cartesian; plane polar (cylindrical); and spherical polar coordinates. In particular:

1. we show how circular and rolling motion can be expressed in Cartesian coordinates; and

2. we show how the kinematical quantities, $\vec{r}$, $\vec{v}$, and $\vec{a}$ are expressed in *curvilinear* coordinates.

This lesson also introduces Newton's three laws of motion, finishing with a discussion on "The audacity of Sir Isaac Newton" to suggest that his three laws apply not just here on Earth, but throughout the universe.

1.2 Cartesian coordinate system (FC §1.10)

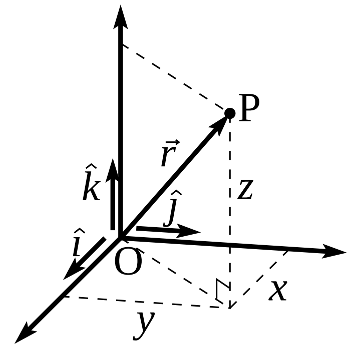
P has displacement $\vec{r}$ from origin, O. Its 3-D *Cartesian coordinates* are:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k};$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + x\frac{d\hat{\imath}}{dt}^{0} + \frac{dy}{dt}\hat{\jmath} + y\frac{d\hat{\jmath}}{dt}^{0} + \frac{dz}{dt}\hat{k} + z\frac{d\hat{k}}{dt}^{0}$$

$$= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k};$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k},$$



Circular Motion

Position of P moving in a circle (radius r, origin O) as a function of time:

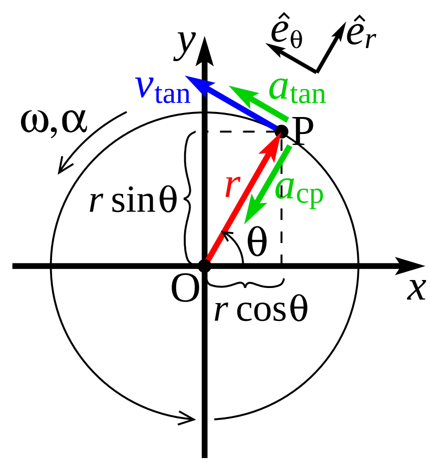
$$\vec{r}(t) = r\cos\theta(t)\,\hat{\imath} + r\sin\theta(t)\,\hat{\jmath}$$

$$= r(\cos\theta, \sin\theta) = r\hat{e}_r,$$

where $\hat{e}_r = (\cos\theta, \sin\theta)$ is parallel to $\vec{r}$ and,

$$|\hat{e}_r| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

Thus, $\hat{e}_r$ is the *radial unit vector.*



For constant r, velocity of P given by:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -r\sin\theta\,\dot{\theta}\,\hat{\imath} + r\cos\theta\,\dot{\theta}\,\hat{\jmath} = \dot{\theta}r(-\sin\theta, \cos\theta) = \underbrace{\omega r}_{v_{\rm tan}}\,\hat{e}_\theta, \quad (1.2.1)$$

where $\hat{e}_\theta = (-\sin\theta, \cos\theta)$ is a unit vector such that,

$$\hat{e}_r \cdot \hat{e}_\theta = (\cos\theta, \sin\theta) \cdot (-\sin\theta, \cos\theta) = 0.$$

Thus, $\hat{e}_\theta \perp \hat{e}_r$ and $\hat{e}_\theta$ is the *tangential unit vector.*

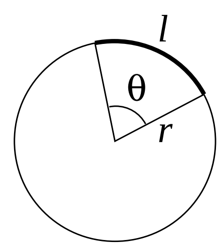
Acceleration of P given by:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -r\cos\theta\,\dot\theta^2\,\hat\imath - r\sin\theta\,\ddot\theta\,\hat\imath - r\sin\theta\,\dot\theta^2\,\hat\jmath + r\cos\theta\,\ddot\theta\,\hat\jmath$$
$$= -\dot\theta^2 r(\cos\theta,\sin\theta) + \ddot\theta r(-\sin\theta,\cos\theta) = -\underbrace{\omega^2 r}_{a_{\rm cp}}\,\hat e_r + \underbrace{\alpha r}_{a_{\rm tan}}\,\hat e_\theta. \quad (1.2.2)$$

In Eq. (1.2.1), (1.2.2),

$$\left.\begin{aligned} \omega &= \dot\theta && = \textit{angular speed};\\ \alpha &= \ddot\theta && = \textit{angular acceleration};\\ v_{\rm tan} &= \omega r && = \textit{tangential velocity};\\ a_{\rm cp} &= \omega^2 r && = \textit{centripetal}\text{ (centre-seeking) }\textit{acceleration};\\ a_{\rm tan} &= \alpha r && = \textit{tangential}\text{ acceleration.} \end{aligned}\right\} \quad (1.2.3)$$

Rolling motion



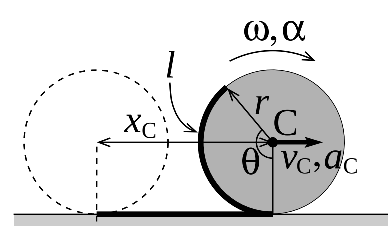
Arc length (portion of circular circumference) given by,

$$l = \theta r, \quad (1.2.4)$$

where r = circle radius, θ (radians) = angle *subtended* by l.

e.g., for $\theta = 2\pi$ (entire circle), $l = 2\pi r$, circle circumference.

To roll without slipping $\Rightarrow x_{\rm C} = l = \theta r$. Thus:



$$v_{\rm C} = \frac{dx_{\rm C}}{dt} = \dot\theta r = \omega r;$$
$$a_{\rm C} = \frac{dv_{\rm C}}{dt} = \ddot\theta r = \alpha r,$$

and *no-slip condition* is:

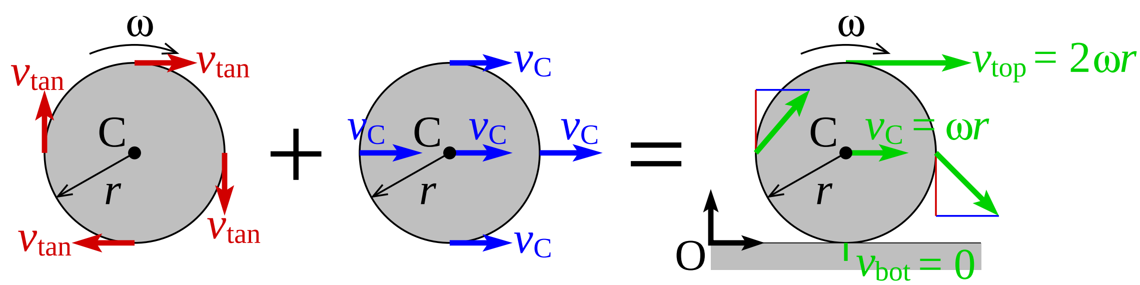
$$\boxed{x_{\rm C} = \theta r; \quad v_{\rm C} = \omega r; \quad a_{\rm C} = \alpha r.} \quad (1.2.5)$$

Comparing Eq. (1.2.5) with Eq. (1.2.3):

- $v_{\rm C}$ (speed of C rel. to ground) = $v_{\rm tan}$ (tangential speed of rim rel. to C);

- a_C (acc$^{\underline{\mathrm{n}}}$ of C rel. to ground) $= a_\mathrm{tan}$ (tangential acc$^{\underline{\mathrm{n}}}$ of rim rel. to C);

Rolling motion = circular + no-slip horizontal motion (vector sum):

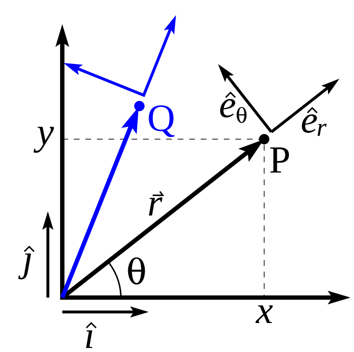


Relative to ground (O):

$$v_\mathrm{top} \;=\; v_\mathrm{tan} + v_\mathrm{C} \;=\; 2\omega r; \qquad v_\mathrm{bot} \;=\; -v_\mathrm{tan} + v_\mathrm{C} \;=\; 0,$$

and *point in contact with ground is momentarily at rest.*

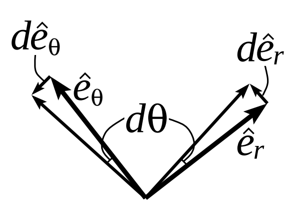
1.3 Curvilinear coordinate systems (FC §1.11, 1.12)



Plane polar coordinates (2-D); (r, θ)

$$\vec{r} \;=\; r\hat{e}_r \;=\; \sqrt{x^2+y^2}\,\hat{e}_r \quad\Rightarrow\quad \vec{v} \;=\; \dot{r}\hat{e}_r + r\dot{\hat{e}}_r.$$

Note that $\dot{\hat{e}}_r \neq 0$; moving from P to Q, $\hat{e}_r$ and $\hat{e}_\theta$ change direction (not magnitude):



$$d\hat{e}_r \;=\; d\theta\,\hat{e}_\theta \qquad\Rightarrow\qquad \dot{\hat{e}}_r \;=\; \dot{\theta}\,\hat{e}_\theta;$$

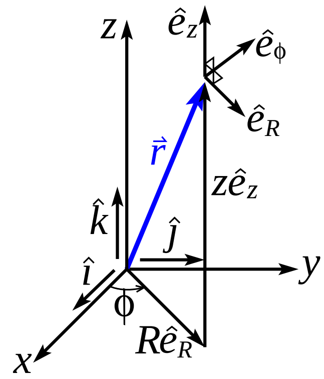
$$d\hat{e}_\theta \;=\; d\theta(-\hat{e}_r) \;\;\Rightarrow\qquad \dot{\hat{e}}_\theta \;=\; -\dot{\theta}\,\hat{e}_r.$$

$$\Rightarrow\quad \vec{v} \;=\; \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \quad \text{and,} \tag{1.3.1}$$

$$\vec{a} \;=\; \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta \;=\; (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta. \tag{1.3.2}$$

Note that Eq. (1.3.1), (1.3.2) reduce to Eq. (1.2.1), (1.2.2) for $r =$ constant.

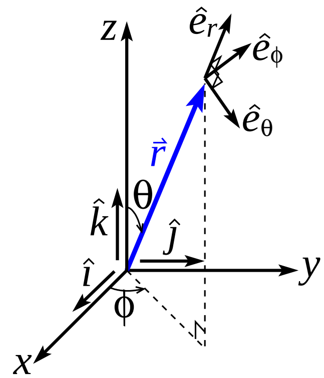
<u>Cylindrical coordinates (3-D); (R,ϕ,z)</u>



= polar coordinates ($r\to R$, $\theta\to\phi$) with a z-axis.

$$\vec{r} = R\hat{e}_R + z\hat{e}_z; \quad (N.B.:\ \hat{e}_z = \hat{k})$$
$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z;$$
$$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z.$$

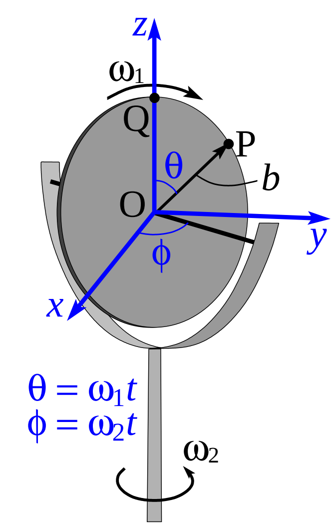
<u>Spherical polar coordinates (3-D); (r,θ,ϕ)</u>



$$\vec{r} = r\hat{e}_r;$$
$$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\sin\theta\hat{e}_\phi;$$
$$\vec{a} = (\ddot{r}\underbrace{-r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2}_{\text{centripetal}})\hat{e}_r$$
$$+ (r\ddot{\theta}\underbrace{-r\dot{\phi}^2\sin\theta\cos\theta}_{\text{centripetal}} + \underbrace{2\dot{r}\dot{\theta}}_{\text{Coriolis}})\hat{e}_\theta + (r\ddot{\phi}\sin\theta + \underbrace{2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta}_{\text{Coriolis}})\hat{e}_\phi.$$

[Last term in equation 1.12.11c (p. 42 ed. 7) should be $\propto \hat{e}_\theta$, not $\hat{e}_\phi$; Eq. 1.12.11c (p. 34, ed. 6) is OK.]

Example 1.1. Find $\vec{a}$ of any point P on rim of wheel.



Solution: Use spherical polar coordinates.

P (relative to O) is at: $(r,\theta,\phi) = (b,\omega_1 t,\omega_2 t)$.

$$\Rightarrow \quad \vec{a}_{\rm P} = (-b\omega_2^2\sin^2(\omega_1 t) - b\omega_1^2)\hat{e}_r$$
$$- b\omega_2^2\sin(\omega_1 t)\cos(\omega_1 t)\hat{e}_\theta + 2b\omega_1\omega_2\cos(\omega_1 t)\hat{e}_\phi,$$

since $\dot{r} = \ddot{r} = \ddot{\theta} = \ddot{\phi} = 0$.

At Q (top of wheel) where $\theta = \omega_1 t = n2\pi$: $\vec{a}_{\rm Q} = -b\omega_1^2\hat{e}_r + 2b\omega_1\omega_2\hat{e}_\phi$.

1.4 Newton's Three Laws of Motion (FC §2.1)

(*Show on-line handout* `newton.pdf`)

Dynamics: Given certain forces acting on a mass m, what is $\vec{r}(t)$?

For this, we use *Newton's three laws of motion*:

1. With no external forces, an object moves with a constant velocity.
2. $\vec{F} = \dot{\vec{p}} = m\vec{a}$ for constant m. $\vec{F} = \sum \vec{F}_{\text{ext}}$, sum of all *external* forces acting on m, $\vec{p} = m\vec{v}$ is linear momentum of m.
3. For every action, there is an equal and opposite reaction. All forces can be identified as a member of an "action-reaction pair".

Newton's second law is a *second order ordinary differential equation*:

$$\vec{F}(\vec{r}, \dot{\vec{r}}, t) = m\frac{d^2\vec{r}}{dt^2}, \tag{1.4.1}$$

which we solve for $\vec{r}(t)$.

<u>Conservation of linear momentum</u>

For no *external forces* on m with momentum $\vec{p}$, then:

$$\vec{F} = \dot{\vec{p}} = 0 \quad \Rightarrow \quad \vec{p} = mv = \text{constant}.$$

For $m = \text{constant}$, $\vec{v} = \text{constant}$ when $\vec{F} = 0$ ($\equiv$ Newton's First Law).

- first law a special case of second law
- introduced as its own law to refute prevailing Aristotelian physics: all motion requires a force to sustain.

Aside: The "audacity" of Sir Isaac Newton

Newton was first to suggest that laws true on Earth apply to whole Universe.

- link made between falling apples and orbiting moon

So let's examine the audacity of this suggestion.

On scale where sun (diameter 1.39×10^9 m) is reduced to a marble (~ 1 cm),

- Earth has diameter 0.1 mm (barely visible dust speck);
- a person is 1.6×10^{-11} m (0.16 Å) tall, about $\frac{1}{6}$ of an atom;
- sun is 1 m away (1.50×10^{11} m);
- nearest star (1.3 pc, 1 pc = 3.3 ly) $\sim$ 300 km (Fredericton);
- centre of galaxy (7,600 pc) $\sim$ 1.7 million km (4.4 $\times$ distance to moon);
- distance to Andromeda galaxy (770,000 pc) $\sim$ 170 million km (> actual distance to sun)
- *and Universe is still 5,000 $\times$ bigger than this!*

And those sub-atomic humans having ventured no further than their dust speck dare to suggest *their* laws apply to this great expanse!!

Tutorial 1.2

Problem 3 (FC 1.17 and 1.20)

a) A moth follows the elliptical path: $\vec{r}(t) = b\cos\omega t\,\hat{\imath} + 2b\sin\omega t\,\hat{\jmath}$ around a light bulb, where b and ω are constant. Using Cartesian coordinates, find the moth's speed, $v(t)$, and then from this find v at $t = 0$ and $t = \pi/(2\omega)$ (when the moth is at the minimum and maximum distances from the light bulb).

b) A fly follows a helical path, given in Cartesian coordinates by:

$$\vec{r}(t) \;=\; b\sin\omega t\,\hat{\imath} + b\cos\omega t\,\hat{\jmath} + ct^2\hat{k},$$

where b, c, and ω are constant. Find $r(t)$, $v(t)$, and $a(t)$ in cylindrical coordinates, and then show that the magnitude of the fly's acceleration, a, is constant.

Lesson 3

This is the first half of the unit on *rectilinear motion*: motion confined to one spatial dimension (1-D). Here we discuss:

1. kinematics,
   - motion with constant acceleration without regard to the responsible forces;
2. assessing forces using,
   - Newton's 2$^{\text{nd}}$ law,
   - *free-body diagrams*;
3. review the *Coulomb model for friction.*

1.5 Rectilinear motion (FC §2.2)

Motion in 1-D ($\vec{r} \to x$) is called *rectilinear motion*, and Eq. (1.4.1) becomes:

$$F(x, \dot{x}, t) = m\ddot{x} = ma. \tag{1.5.1}$$

1.5.1 Kinematics

$\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = a \;\Rightarrow\; dv = adt$. For a constant (and *only* for a constant),

$$\int dv = a\int dt \quad\Rightarrow\quad v(t) = v_0 + at, \tag{1.5.2}$$

where v_0 is a constant of integration. Integrating again, we get:

$$\int v\,dt = \int \frac{dx}{\cancel{dt}}\cancel{dt} = \int (v_0 + at)dt \quad\Rightarrow\quad x(t) = x_0 + v_0 t + \tfrac{1}{2}at^2, \tag{1.5.3}$$

where x_0 is a constant of integration. Eliminate t between Eq. (1.5.2), (1.5.3):

$$\Rightarrow \quad v^2(x) = v_0^2 + 2a(x - x_0). \tag{1.5.4}$$

Eq. (1.5.2)–(1.5.4) are equations of 1-D kinematics, *valid only for constant a*.

Example 1.2. Drop a stone from rest and height h. Find v when it lands.

Solution: Here, there's no mention of t, so we'll use Eq. (1.5.4)

Stone falls from $x_0 = h$ to $x = 0$ with constant acceleration $-g$ (downward)[1].

$$\Rightarrow \quad v^2 = \cancelto{0}{v_0^2} + 2(-g)(0 - h) = 2gh \quad\Rightarrow\quad \boxed{v = \sqrt{2gh}.}$$

Note how negative signs cancel so that $v^2 > 0$, as it *must* be. □

[1] "Downward" is negative because "upward" has already been defined as positive ($x = +h$).

Example 1.3. A car is stopped at a red light. As light turns green, a truck moving at constant v passes the car, which accelerates at constant a. Where and when does car overtake truck, and what is car's speed?

Solution: Let position of car and truck be x_c and x_t; let initial position of car be $x_0 = 0$. Then from Eq. (1.5.3),

$$x_\mathrm{c} = \cancelto{0}{x_0} + \cancelto{0}{v_0}t + \frac{1}{2}at^2 = \frac{1}{2}at^2; \qquad x_\mathrm{t} = \cancelto{0}{x_0} + vt = vt,$$

since acceleration of truck is zero. Thus, for car to overtake truck,

$$x_\mathrm{c} = x_\mathrm{t} \quad \Rightarrow \quad \frac{1}{2}at^2 = vt \quad \Rightarrow \quad t = 0 \quad \text{or} \quad \boxed{\frac{2v}{a}.}$$

Thus, $x_\mathrm{c} = x_\mathrm{t} = vt = \boxed{2v^2/a.}$ ($t = 0$ is when truck passed car.)

Finally, from Eq. (1.5.4),

$$v_\mathrm{c}^2 = \cancelto{0}{v_0^2} + 2a(x_\mathrm{c} - \cancelto{0}{x_0}) = 2\cancel{a}\frac{2v^2}{\cancel{a}} = 4v^2 \quad \Rightarrow \quad v_\mathrm{c} = 2v \qquad \square$$

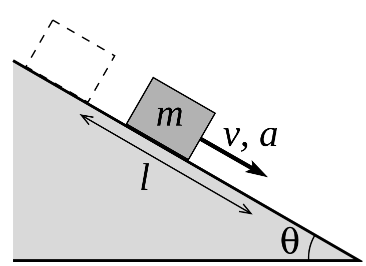
1.5.2 Applying Newton's Second Law

Write Newton's second law as:

$$\sum \vec{F} = m\vec{a}, \tag{1.5.5}$$

a *vector equation*, where $\vec{a}$ need not be constant.

Example 1.4. A block of mass, m, slides down a plane inclined at angle θ, with coefficient of kinetic friction μ_k.

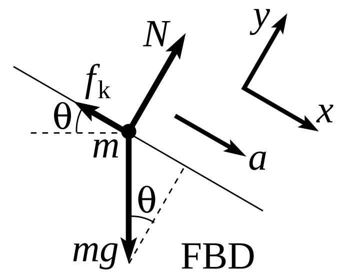


a) Find a.

b) Starting from rest, find v after m slides a distance l.

Solution: a) Here, we demonstrate use of a *free-body diagram* (FBD).

Step 1: Identify relevant forces, construct FBD



- mg, N, $f_{\rm k}$

- reduce m to a dot (centre of mass, CM)

- anchor forces to dot, label with magnitudes

- forces point in direction of action, if known (else, align with + axes)

- $\vec{a}$ (and/or $\vec{v}$) included "off-dot"

- coordinate system defined (aligned with $\vec{a}$ and/or plurality of $\vec{F}$)

Step 2: Break vector equation Eq. (1.5.5) into components:

$$x/ \quad \sum F_x = ma_x \quad \Rightarrow \quad mg\sin\theta - f_{\rm k} = ma; \tag{1.5.6}$$

$$y/ \quad \sum F_y = ma_y \quad \Rightarrow \quad N - mg\cos\theta = 0 \quad \Rightarrow \quad N = mg\cos\theta. \tag{1.5.7}$$

Note that;

- N does not always equal mg (as in this example);

- no vector symbols ($\vec{\ }\,\hat{\ }$) in component equations.

- a_x and a_y are different; don't use same symbol for each (*e.g.*, a)!

Step 3: Include physical models or constraints, if any:

$$f_{\rm k} = \mu_{\rm k} N. \tag{1.5.8}$$

Step 4: Perform algebra. Substitute Eq. (1.5.7) into (1.5.8):

$$f_k = \mu_k mg\cos\theta, \tag{1.5.9}$$

and then Eq. (1.5.9) into (1.5.6):

$$mg(\sin\theta - \mu_k\cos\theta) = ma$$

$$\Rightarrow \quad a = g(\sin\theta - \mu_k\cos\theta) = g\cos\theta(\tan\theta - \mu_k). \tag{1.5.10}$$

$$\text{Note that if } \begin{cases} \mu_k < \tan\theta, & a > 0 \ (m \text{ speeds up as it goes downhill}); \\ \mu_k = \tan\theta, & a = 0 \ (m \text{ moves with constant speed}); \\ \mu_k > \tan\theta, & a < 0 \ (m \text{ slows down and eventually stops}). \end{cases}$$

"Critical angle" $\theta_k = \tan^{-1}\mu_k$ is *angle of kinetic friction.*

b) Since a is constant, use kinematics; since t not mentioned, use Eq. (1.5.4):

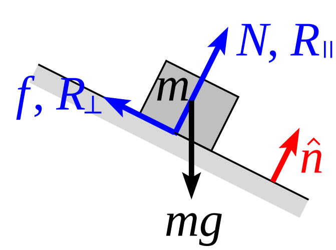
$$v^2 = \cancelto{0,\text{ starts from rest}}{v_0^2} + 2a(x - x_0) = 2g\cos\theta(\tan\theta - \mu_k)l \quad (\text{using Eq. 1.5.10}).$$

$$\Rightarrow \quad \boxed{v = \sqrt{2gl\cos\theta(\tan\theta - \mu_k)},} \tag{1.5.11}$$

which makes sense ($v \in \mathbb{R}$) only if $\mu_k < \tan\theta$ (and thus $\theta > \theta_k$).

Aside: Coulomb model *for static and kinetic friction*

(*Show on-line handout* `friction.pdf`)



Friction force between two surfaces given by,

$$f = \begin{cases} f_s \le \mu_s N, & v_{rel} = 0 \ (\text{static friction}); \\ f_k = \mu_k N, & v_{rel} > 0 \ (\text{kinetic friction}), \end{cases} \tag{1.5.12}$$

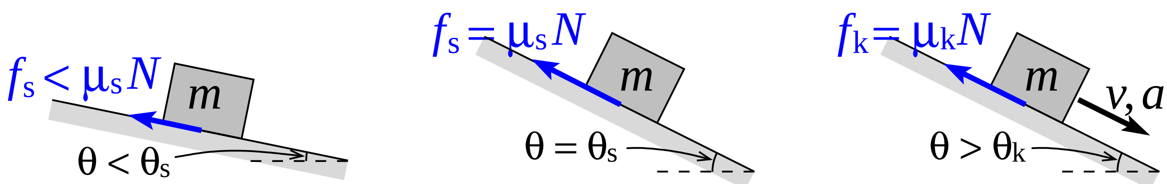
N = normal force between surfaces, μ_s (μ_k) is *coeff. of static* (*kinetic*) *friction.*

- Normal force and friction are *surface forces*, often written as $N = R_\parallel \parallel \hat{n}$, $f = R_\perp \perp \hat{n}$; $\hat{n}$ is a unit vector pointing in direction normal to surface.

- Direction of f_k opposite to motion of surface.

- Direction of f_s opposite to motion surface would have in its absence.

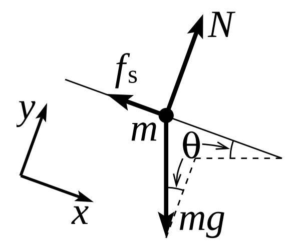
- $f_\text{s} = \mu_\text{s} N$ only when surface is on *verge of slipping*. Otherwise, $f_\text{s} < \mu_\text{s} N$.



To find *angle of repose* (θ_s, where $f_\text{s} = \mu_\text{s} N$), consider FBD for $v = a = 0$:

$$x/\quad mg\sin\theta - f_\text{s} \;=\; 0; \qquad y/\quad N - mg\cos\theta \;=\; 0$$

$$\Rightarrow \quad f_\text{s} \;=\; mg\sin\theta \quad \text{and} \quad N \;=\; mg\cos\theta.$$



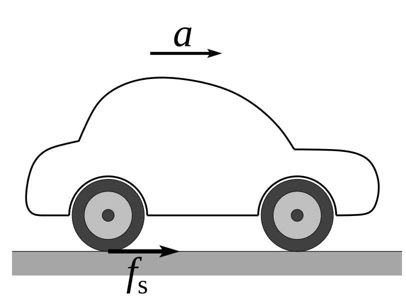
For maximum f_s, set $f_\text{s} = \mu_\text{s} N$:

$$\cancel{mg}\sin\theta_\text{s} \;=\; \mu_\text{s}\cancel{mg}\cos\theta_\text{s} \quad \Rightarrow \quad \theta_\text{s} \;=\; \tan^{-1}\mu_\text{s} \;>\; \tan^{-1}\mu_\text{k} \;=\; \theta_\text{k},$$

angle of kinetic friction (defined after Eq. 1.5.10).

Example 1.5. A car accelerates from rest without "squealing out". What type of friction acts between tires and road, and what is its direction?

Solution: Tires *roll without slipping.*



- point of tire in contact with road always at rest even as car moves forward.

$\Rightarrow$ <u>friction is *static* friction.</u>

If tire slips, bottom of tire moves backward relative to road.

- This is motion f_s is resisting (not motion of car)

$\Rightarrow$ f_s points forward.

Force exerted by engine is *not* what accelerates car (not directly).

- With no friction between tires and road, wheels spin and car goes nowhere regardless of what engine does.
- Engine applies force to axle (not road!) and is an *internal* force to the car; only *external* forces can accelerate car.

Force that accelerates car is friction delivered by the road to the tires.

- Since car accelerates forward, f_s *must* point forward (as already shown).

Tutorial 1.3

Problem 4 (FC 2.10) A block of mass m is shoved up a sloped plane with initial velocity v_0. If the inclination angle of the plane is $\theta = 30°$ and the coefficient of kinetic friction between the block and plane is $\mu_k = 0.1$, find in terms of v_0 how long it takes for the block to slide back down to where it started. You may assume that the static coefficient of friction is insufficient to stop the block at the top of its rise.

Lesson 4

This is the second half of the unit on *rectilinear motion*: motion confined to one spatial dimension (1-D). Here we:

1. continue our discussion on free-body diagrams with friction;

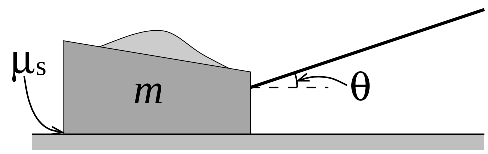
2. learn to assess forces using Newton's 3^{rd} law;

3. go through a few examples in detail.

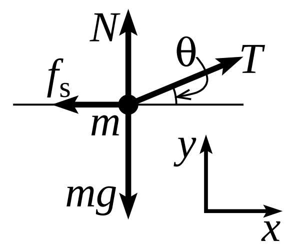
Example 1.6. A box of sand, m, is dragged across a floor by a cable rated for a maximum tension of 1,100 N. If μ_s between box and floor is 0.35, find:

a) θ at which maximum m is dragged;

b) maximum mass of sand + box, m_{max}.



Step 1: Construct FBD for m for $a = 0$, v = constant.



Forces are "used up" in two ways:

1. balancing other forces;
2. accelerating masses.

To maximise m, don't use any of T to accelerate m; thus set $a = 0$.

Step 2: Break vector equation Eq. (1.5.5) into components:

$$x/ \quad \sum F_x = ma_x \quad \Rightarrow \quad -f_s + T\cos\theta = 0; \tag{1.5.13}$$

$$y/ \quad \sum F_y = ma_y \quad \Rightarrow \quad N + T\sin\theta - mg = 0. \tag{1.5.14}$$

Step 3: Physical models/constraints: T must withstand *maximum* f_s,

$$\Rightarrow \quad f_s = \mu_s N = \mu_s(mg - T\sin\theta), \tag{1.5.15}$$

using Eq. (1.5.14).

Step 4: Perform algebra. Substitute Eq. (1.5.15) into (1.5.13):

$$T\cos\theta = \mu_s(mg - T\sin\theta) \quad \Rightarrow \quad m(\theta) = \frac{T}{g}\left(\frac{\cos\theta}{\mu_s} + \sin\theta\right) \tag{1.5.16}$$

To find θ that maximises m,

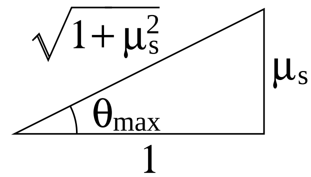
$$\frac{dm}{d\theta} = \frac{T}{g}\left(-\frac{\sin\theta}{\mu_s} + \cos\theta\right) = 0 \quad \Rightarrow \quad \tan\theta = \mu_s$$

a) Thus, angle to maximise dragged m:

$$\theta_{\max} = \tan^{-1}\mu_s \sim 0.337\,\text{rad} \sim \underline{\underline{19.3°}}.$$

b) If $\tan\theta_{\max} = \mu_s = \frac{\mu_s}{1}$, then from right triangle,

$$\cos\theta_{\max} = \frac{1}{\sqrt{1+\mu_s^2}} \quad \text{and} \quad \sin\theta_{\max} = \frac{\mu_s}{\sqrt{1+\mu_s^2}}.$$



Substituting these into Eq. (1.5.16), we get:

$$m_{\max} = \frac{T}{g}\left(\frac{1}{\mu_s}\frac{1}{\sqrt{1+\mu_s^2}} + \frac{\mu_s}{\sqrt{1+\mu_s^2}}\right) = \frac{T}{\mu_s g}\sqrt{1+\mu_s^2} \sim \underline{\underline{339\,\text{kg}}}.$$

Note that $m(0) = \frac{T}{\mu_s g} \sim 320\,\text{kg}$. $\theta = 19.3°$ allows N and thus f_s to be less.

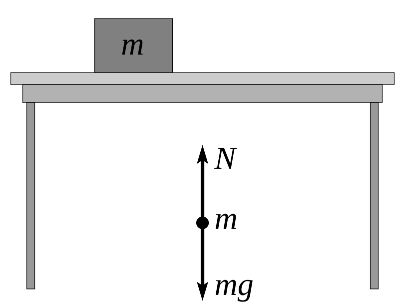
1.5.3 Applying Newton's Third Law

Teaser: *Since for every action there's an equal and opposite reaction, action-reaction forces cancel, leaving no net force. Thus, objects can't accelerate?!?*

Action-reaction pairs *always*...

1. have equal magnitude, opposite direction;
2. act between two objects and are the same *type* (*e.g.*, mg, N, *etc.*);
3. act on different objects and thus *never* appear on same FBD.

Example 1.7. An object is at rest on a table. If "action-force" is mg, what is "reaction force"?



Most—including some physics professors—will say N (table on m). **Wrong!!**

- N is an electromagnetic force, mg gravity; different type! (rule 2)

- N and mg act on same object and thus on same FBD! (rule 3)

To identify an action-reaction pair:

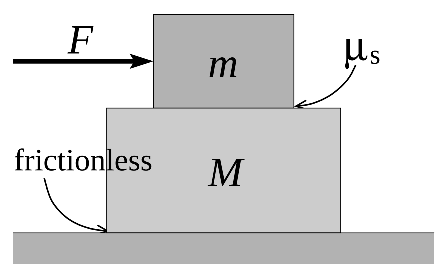
1. describe "action" as: *object A exerts a force of type X on object B*;

2. swap A and B (and then reverse direction) to get "reaction": *object B exerts a force of (same) type X on object A.*

Action force: Earth exerts gravitational force, mg, downward on m.
Reaction force: m exerts *gravitational* force, mg, upward on Earth.

Action force: table exerts a normal force, N, upward on m.
Reaction force: m exerts a normal force, N, downward on table.

Example 1.8. F is applied to m sitting on M. M sits on a frictionless surface; coefficient of static friction between m and M is μ_s.



For what F will m start to slip?

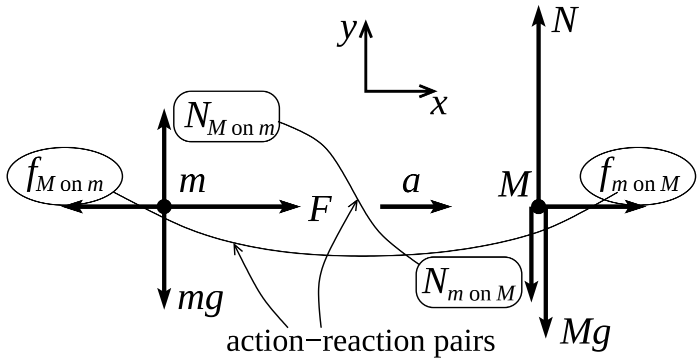
Solution: Here, we must consider static friction, f_s, between m and M.

- Only external forces are labelled on an FBD.

- If we consider m and M as one object, f_s is an internal force.

- Consider FBDs for m and M separately so that f_s is external to each.

Step 1: Construct FBDs.



From Newton's 3rd Law:

$$f_{M\,\text{on}\,m} = f_{m\,\text{on}\,M} \equiv f_\text{s};$$

$$N_{M\,\text{on}\,m} = N_{m\,\text{on}\,M} \equiv N_{Mm}.$$

Step 2: Break $\sum \vec{F} = m\vec{a}$ into components for each mass.

$$\text{for } m: \quad x/ \ \sum F_x = ma_x \quad \Rightarrow \quad F - f_\text{s} = ma; \tag{1.5.17}$$

$$y/ \ \sum F_y = ma_y \quad \Rightarrow \quad N_{Mm} - mg = 0; \tag{1.5.18}$$

$$\text{for } M: \quad x/ \ \sum F_x = Ma_x \quad \Rightarrow \quad f_\text{s} = Ma; \tag{1.5.19}$$

$$y/ \ \sum F_y = Ma_y \quad \Rightarrow \quad N - Mg - N_{Mm} = 0 \tag{1.5.20}$$

Note: F acts on m, not M!

Step 3: Models/constraints. For maximum F, use maximum f_s:

$$f_\text{s,max} = \mu_\text{s} N_{Mm} = \mu_\text{s} mg, \tag{1.5.21}$$

using Eq. (1.5.18).

Step 4: Do the algebra. $M \times$ Eq. (1.5.17) $- m \times$ Eq. (1.5.19) gives:

$$M(F - f_s) - mf_s = 0 \quad \Rightarrow \quad F = \frac{M+m}{M} f_\text{s}$$

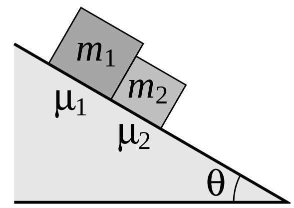
$$\Rightarrow \quad \boxed{F_\text{max} = \frac{M+m}{M} \mu_\text{s} mg,}$$

using Eq. (1.5.21). Note that Eq. (1.5.20) was never used (N not needed).

Exercise: Find maximum acceleration.

Tutorial 1.4

Problem 4 Two blocks of masses m_1 and m_2 remain in contact as they slide down a ramp inclined at an angle θ to the horizontal. The coefficient of kinetic friction between each of the blocks and the ramp is μ_1 and μ_2 respectively.



a) What is the normal force acting between the two masses and what is the criterion that it be greater than zero?

b) What is their common acceleration down the ramp?

c) If $\mu_2 = 2\mu_1$ and $m_2 = \frac{1}{2}m_1$, what is the maximum value for μ_1 if $\theta = 30°$?

Lesson 5

This is the first of three lessons in solving *differential equations*. Here we:

1. define both *ordinary* and *partial* differential equations (ODEs, PDEs);

2. make the link between solving an ODE and integration;

3. learn to solve simple *first-order* ODEs by,

    - *inspection*,

    - *separation of variables*.

1.6 First lesson in solving ODEs

Ordinary differential equation (ODE) has derivatives of *univariate* functions:

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) \;=\; F.$$

Partial differential equation (PDE) has derivatives of *multivariate* functions:

$$-\frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \;=\; i\hbar\frac{\partial\psi(x,t)}{\partial t}.$$

Equation whose highest order derivative is n is an n^{th} *order ODE* (*PDE*).

- PHYS 2302/2303: simplest types of 1^{st} and 2^{nd} order ODEs
- MATH 2303: *any* 1^{st} order ODE; 2^{nd} order ODEs with constant coefficients
- PHYS 3200: *any* 2^{nd} order ODE; *separable* 2^{nd} order PDEs

Consider the two equations:

$$\frac{dy}{dx} \;=\; ay \qquad \text{and} \qquad x \;=\; \frac{1}{a}\int\frac{dy}{y}, \tag{1.6.1}$$

a first order ODE[2] and an integral equation.

These are the same equation!

$$\frac{dy}{dx} = ay \quad\Rightarrow\quad \frac{dy}{y} = a\,dx \quad\Rightarrow\quad \int\frac{dy}{y} = \int a\,dx = ax \quad\Rightarrow\quad x = \frac{1}{a}\int\frac{dy}{y}.$$

To find $y(x)$, we "do the integral" and get:

$$x \;=\; \frac{1}{a}\ln y + c \quad\Rightarrow\quad \ln y \;=\; a(x-c) \quad\Rightarrow\quad y(x) \;=\; e^{a(x-c)} \;=\; Ce^{ax},$$

[2] *Pssst*! We already solved *two* 1^{st} order ODEs in §1.5.1 (kinematics)!

where $C = e^{-ac}$ is a constant set from a boundary condition. So, if $y = y_0$ at $x = 0$,

$$y(0) = Ce^{a(0)} = C = y_0 \quad \Rightarrow \quad y(x) = y_0 e^{ax},$$

is the final solution.

Both Eq. (1.6.1) are solved in "direction of antidifferentiation":

direction of antidifferentiation (integration) $\longrightarrow$

$$\cdots \quad \frac{d^2f(x)}{dx^2} \quad \frac{df(x)}{dx} \quad f(x) \quad \int^x f(y)dy \quad \int^x\!\!\int^y f(z)dzdy \quad \cdots$$

$\longleftarrow$ direction of differentiation

$$\frac{dy}{dx} \to y(x) \qquad \text{and} \qquad g(y) = \frac{1}{y} \to \int g(y)dy.$$

And there's the rub! *There's no algebraic way to antidifferentiate*!

Direction of differentiation is different. Given $f(x)$, we can *always* use definition of a derivative to find $f'(x)$ algebraically.

Example 1.9. $f(x) = \cos x$. Then,

$$\frac{df}{dx} = \lim_{\delta x \to 0} \frac{\cos(x+\delta x) - \cos x}{\delta x} = \lim_{\delta x \to 0} \frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x}{\delta x},$$

using a trig identity. Then, for $\delta x \to 0$, $\cos \delta x \to 1$ and $\sin \delta x \to \delta x$,

$$\Rightarrow \quad \frac{df}{dx} = \lim_{\delta x \to 0} \frac{\cancel{\cos x} - \sin x\, \cancel{\delta x} - \cancel{\cos x}}{\cancel{\delta x}} = -\sin x.$$

Example 1.10. $f(x) = \ln x$. Then,

$$\begin{aligned}\frac{df}{dx} &= \lim_{\delta x \to 0} \frac{\ln(x+\delta x) - \ln x}{\delta x} = \lim_{\delta x \to 0} \frac{\ln\left[x(1+\delta x/x)\right] - \ln x}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\cancel{\ln x} + \ln(1+\delta x/x) - \cancel{\ln x}}{\delta x} = \lim_{\delta x \to 0} \frac{\cancel{\delta x}/x}{\cancel{\delta x}} = \frac{1}{x},\end{aligned}$$

since $\ln(1+\epsilon) = \epsilon + \dfrac{\epsilon^2}{2} + \dfrac{\epsilon^3}{3} + \cdots \to \epsilon$ for $\epsilon \ll 1$.

One may need to know trig identities or expansions, but one can always find a derivative algebraically from definition of derivative.

Not so for antidifferentiation. Whether integrating or solving an ODE, there always comes a time when one "simply has to recognise the solution".

All methods of integration or solving ODEs boil down to beating equation into a recognisable form, then writing down the answer.

In solving Eq. (1.6.1), we beat the ODE algebraically to:

$$x = \frac{1}{a}\int \frac{dy}{y},$$

at which point we "did the integral" by recognising it and writing down:

$$x = \frac{1}{a}\ln y + c,$$

then continued with the algebra.

In PHYS 2302, we shall limit ourselves to:

1^{st} order ODEs that can be:

- easily converted to an integral; or
- solved directly by "inspection",

2^{nd} order ODEs that can be:

- converted to two first order ODEs (then see above); or
- solved directly by "inspection"; or
- solved by trial exponential solutions.

Example 1.11. Solve $\dfrac{dy}{dx} = ay$ again, this time by "inspection".

Solution: For simple ODEs such as this, ask:

What function, $y(x)$, has a first derivative equal to itself times a constant?

Dropping "times a constant", the answer is easy: e^x is the only function whose derivative is itself:

$$\frac{de^x}{dx} = e^x.$$

So, if we let $y = e^x$, we'd have $\dfrac{dy}{dx} = y$; we want ay.

After further inspection, try $y = e^{ax}$ so that by the chain rule,

$$\frac{dy}{dx} = ae^{ax} = ay,$$

as desired.

But wait! Ce^{ax} (C any constant) still solves the ODE:

$$\frac{dy}{dx} = \frac{d}{dx}Ce^{ax} = C\frac{d}{dx}e^{ax} = Cae^{ax} = aCe^{ax} = ay.$$

Thus, $y(x) = Ce^{ax}$ is the general solution, as found before by integration.

Whether by integration or inspection, there's always an "alakazam moment" where something is simply recognised.

- integration may seem a little more elegant, but it's usually longer

- inspection may seem daunting, but if you can see it, go for it!

Method used to convert ODE to an integral is called *separation of variables*.

Starting with,

$$\frac{dy}{dx} = ay,$$

put all occurrences of y on one side of equation, x on the other:

$$\frac{dy}{y} = a\,dx.$$

Once in this form, integrate both sides and solve.

Not all first order ODEs can be separated, *e.g.* $\frac{dy}{dx} = y + x$. (Try it!)

Those that can are said to be *separable.*

General form for a separable first order ODE is:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}, \tag{1.6.2}$$

where $f(x)$ and $g(y)$ are arbitrary functions. In this form, we write:

$$g(y)dy = f(x)dx \quad \Rightarrow \quad \int g(y)dy = \int f(x)dx,$$

then, "do the integrals", then solve for $y(x)$ algebraically.

We'll come across numerous examples in the physics.

Tutorial 2.1

Problem 1 Solve each of the first order ODEs for $y(x)$ using separation of variables:

a) $4x\frac{dy}{dx} - y(x-3) = 0;$

b) $\cos x\frac{dy}{dx} + y\sin x = 0;$

c) $(x^2-4)\frac{dy}{dx} + y = -1.$

Lesson 6

Here, we apply what we learned in the previous lesson to solve simple first order ODEs arising in problems involving:

1. forces in 1-D dependent only upon position;
2. forces in 1-D dependent only upon velocity.

In so doing, we uncover important principles and concepts in physics such as:

1. the *work-kinetic (WK) theorem*;
2. *conservation of mechanical energy*;
3. *characteristic time scales*;
4. *terminal velocity.*

1.7 Forces dependent only upon position (FC §2.3)

Such forces include:

- gravity, $-\dfrac{GMm}{r^2}\hat{e}_r$; $-mg\hat{k}$ near surface of Earth;
- Hooke's law, $-kx$, where x is distortion of spring,

but do not include:

- friction, $\mu_k N$, also depends upon direction of motion;
- string tension, T, ditto.

Suppose $F(x, \dot{x}, t) = F(x)$. Then Eq. (1.5.1) can be written:

$$F(x) = m\ddot{x} = m\frac{d^2x}{dt^2}, \tag{1.7.1}$$

a 2^{nd} order ODE to solve for $x(t)$ with a given $F(x)$.

Use *chain rule* to split Eq. (1.7.1) into two 1^{st} order ODEs. Thus,

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \frac{dv}{dx}v, \tag{1.7.2}$$

where $v = \dot{x}$. Then, Eq. (1.7.1) becomes:

$$F(x) = mv\frac{dv}{dx}, \tag{1.7.3}$$

a 1^{st} order ODE (separable); whose solution is $v(x)$ for a given $F(x)$.

In deriving Eq. (1.7.3), we used:

$$v(x) = \dot{x} = \frac{dx}{dt}, \tag{1.7.4}$$

a second 1^{st} order ODE (separable) whose solution gives $x(t)$ for a given $v(x)$.

Next, separate Eq. (1.7.3), $F(x)dx = mv\,dv$, then integrate from $x_0 \to x$:

$$\int_{x_0}^{x} F(x')dx' = m\int_{v_0}^{v} v'dv' = \tfrac{1}{2}mv'^2\Big|_{v_0}^{v} = \tfrac{1}{2}m(v^2 - v_0^2), \qquad (1.7.5)$$

where $v_0 = v(x_0)$. Now, define:

$$W = \int_{x_0}^{x} F(x')dx' \qquad \text{and} \qquad K = \tfrac{1}{2}mv^2, \qquad (1.7.6)$$

as *work* done by external forces $F(x)$ over distance $x - x_0$, and *kinetic energy* of m[3]. Then, Eq. (1.7.5) becomes:

$$\boxed{W = K - K_0,} \qquad (1.7.7)$$

where $K_0 = K$ at $x = x_0$. This is the *work-kinetic* (W-K) *theorem*:

$$\boxed{\textit{Work done by all external forces acting on } m \textit{ is its change in KE,}}$$

which we'll return to when we derive it in 3-D (Chapter 4).

Next, and only because $F = F(x)$, define *potential energy* of m, $U(x)$ as:

$$F(x) = -\frac{dU(x)}{dx} \quad \Rightarrow \quad F(x)dx = -dU$$

$$\Rightarrow \quad W = \int_{x_0}^{x} F(x')dx' = -\int_{U_0}^{U} dU' = -U + U_0, \qquad (1.7.8)$$

where $U_0 = U(x_0)$. Comparing Eq. (1.7.7), (1.7.8), we get:

$$K - K_0 = -U + U_0 \quad \Rightarrow \quad K + U = K_0 + U_0 = \text{constant} \equiv E,$$

where $E = K + U$ is the *mechanical energy* of the system:

$$\boxed{\textit{When } F = F(x), \textit{ mechanical energy is conserved.}}$$

[3]Fowles and Cassiday use what most outside the UK would consider non-standard nomenclature for many of their physical variables. Thus, they use T for kinetic energy, not K, τ for period, not T, N for torque, not τ, R for normal force, not (always) N, *etc.* I shall not follow their convention.

Thus, $K = \frac{1}{2}mv^2 = E - U(x)$,

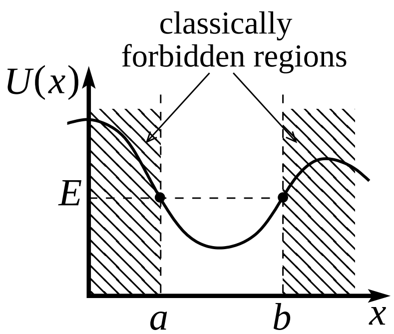
$$\Rightarrow \quad v(x) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}, \tag{1.7.9}$$

solves Eq. (1.7.3), with $U(x)$ given by Eq. (1.7.8) for a given $F(x)$.

We now separate Eq. (1.7.4): $dt = \dfrac{dx}{v(x)}$, then integrate from $t_0 \to t$:

$$\Rightarrow \quad \int_{t_0}^{t} dt' = t - t_0 = \int_{x_0}^{x} \frac{dx'}{v(x')} = \pm\int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}\big(E - U(x')\big)}}, \tag{1.7.10}$$

which, in principle, can be solved for $x(t)$, final solution to Eq. (1.7.4), (1.7.1).



Eq. (1.7.9), (1.7.10) are real only if $U(x) \leq E$. Particle with mechanical energy E is confined to region between a and b (*turning points*); the *classically allowed region*. In Quantum Mechanics, particles can enter "forbidden regions".

Example 1.12. *Escape velocity*: Ignoring air resistance, find minimum v_0 so that projectile, m, launched from Earth's surface ($r = R_\oplus$) doesn't return.

Solution: Earth's gravity, $F(r) = -\dfrac{GM_\oplus m}{r^2}$ ($< 0 \Rightarrow$ attractive force), is a position-dependent force. Thus, from Eq. (1.7.8),

$$\begin{aligned} -U(r) + U(R_\oplus) &= \int_{R_\oplus}^{r} F(r')dr' = -GM_\oplus m \int_{R_\oplus}^{r} \frac{1}{r'^2}dr' \\ &= GM_\oplus m\left(\frac{1}{r} - \frac{1}{R_\oplus}\right) = K(r) - K(R_\oplus). \end{aligned}$$

Minimum $v_0 \Rightarrow v \to 0$ (turning point) as $r \to \infty$. Thus,

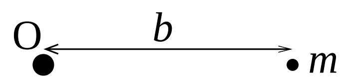
$$GM_\oplus \cancel{m}\Big(\overset{\nearrow 0}{\cancel{\tfrac{1}{r}}} - \frac{1}{R_\oplus}\Big) = \frac{1}{2}\cancel{m}\overset{\nearrow 0}{\cancel{v^2}} - \frac{1}{2}\cancel{m}v_0^2 \quad \Rightarrow \quad v_0 = \sqrt{\frac{2GM_\oplus}{R_\oplus}},$$

is the escape velocity. Now, on Earth's surface,

$$\cancel{m}g = \frac{GM_\oplus \cancel{m}}{R_\oplus^2} \quad \Rightarrow \quad GM_\oplus = gR_\oplus^2 \quad \Rightarrow \quad v_0 = v_{\text{esc}} = \sqrt{2gR_\oplus}.$$

For $g = 9.81\,\text{m}\,\text{s}^{-2}$ and $R_\oplus = 6.37 \times 10^6\,\text{m}$, $v_{\text{esc}} \sim 11.2\,\text{km}\,\text{s}^{-1}$.

Example 1.13. (Problem 2.14): m is released from rest at distance b from O that attracts m with force: $F(x) = -\dfrac{k}{x^2}$. Find time for m to reach O.



Solution: From Eq. (1.7.8),

$$-U(x) + U(b) = -k\int_b^x \frac{1}{x'^2}dx' = \frac{k}{x} - \frac{k}{b} \quad \Rightarrow \quad U(x) = -\frac{k}{x}.$$

E is constant and can be evaluated anywhere. Thus, at $x = b$:

$$E = \cancelto{0}{K(b)} + U(b) = -\frac{k}{b},$$

and Eq. (1.7.10) becomes:

$$t = \int_b^0 \left[\frac{2}{m}\left(\frac{k}{x} - \frac{k}{b}\right)\right]^{-1/2} dx = \sqrt{\frac{mb}{2k}} \int_b^0 \sqrt{\frac{x}{b-x}}\,dx.$$

Let $x = b\sin^2\phi \Rightarrow dx = 2b\sin\phi\cos\phi\,d\phi$. Then, at $x = b$, $\sin^2\phi = 1 \Rightarrow \phi = \pi/2$; at $x = 0$, $\sin^2\phi = 0 \Rightarrow \phi = \pi$[4]. Thus,

$$t = \sqrt{\frac{mb}{2k}}\,2b\int_{\pi/2}^{\pi} \frac{\sin\phi}{\cancel{\sqrt{1-\sin^2\phi}}} \sin\phi \cancel{\cos\phi}\,d\phi = \sqrt{\frac{2mb^3}{k}} \int_{\pi/2}^{\pi} \sin^2\phi\,d\phi$$

$$= \sqrt{\frac{2mb^3}{k}} \int_{\pi/2}^{\pi} \frac{1-\cos 2\phi}{2}\,d\phi = \sqrt{\frac{mb^3}{2k}}\left(\phi - \tfrac{1}{2}\sin 2\phi\right)\Big|_{\pi/2}^{\pi} = \sqrt{\frac{mb^3}{2k}}\frac{\pi}{2}.$$

$$\Rightarrow \quad \boxed{t = \pi\sqrt{\frac{mb^3}{8k}}.}$$

See also examples 2.3.1 – 2.3.5 in text.

[4]Indeed, at $x = b$, $\sin\phi$ could have used any of $\pi/2 + n\pi$, $n \in \mathbb{Z}$. Similarly, at $x = 0$, $\sin\phi$ could have used any of $n\pi$. We chose two that would give the time to get from b to O (1/4 of an "orbit"). Choosing $\pi/2 \to 0$ would have been the time m took to get from O to b. Choosing $\pi/2 \to 2\pi$ would be the time for m to fall through O, rise to $-b$, then fall back to O, *etc.*

1.8 Forces dependent only upon velocity (FC §2.4)

For $F(x, v, t) = F(v)$, Eq. (1.5.1) can be written:

$$F(v) = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m\frac{dv}{dx}v,$$

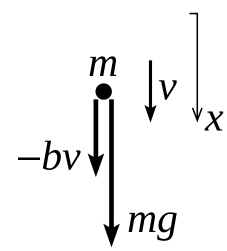
using chain rule again. This yields two possible separable 1st order ODEs:

$$dt = m\frac{dv}{F(v)} \quad \Rightarrow \quad \int_{t_0}^{t} dt' = t - t_0 = m\int_{v_0}^{v} \frac{dv'}{F(v')}; \tag{1.8.1}$$

$$dx = mv\frac{dv}{F(v)} \quad \Rightarrow \quad \int_{x_0}^{x} dx' = x - x_0 = m\int_{v_0}^{v} \frac{v'dv'}{F(v')}, \tag{1.8.2}$$

where $x(t_0) = x_0$, $v(t_0) = v_0$.

Example 1.14. A mass m falling from rest experiences a "linear" air drag $F_\mathrm{d} = -bv$. Find $v(t)$ and $x(t)$.



Solution: In FBD[5], $+x$, v point in same direction (down).

- if x were up and v down, we'd need to use $v = -\dfrac{dx}{dt}$.
- if v were upward, then $v < 0$.
- drag force pointing in $+x$ direction labeled with minus sign: $-bv$.

Thus, in x-direction, $F(v) = mg + (-bv)$. Then, from Eq. (1.8.1) with $t_0 = 0$, $v_0 = 0$:

$$t = m\int_0^v \frac{dv'}{mg - bv'} = \frac{m}{b}\int_0^v \frac{dv'}{v_\mathrm{t} - v'} = -\frac{m}{b}\ln(v_\mathrm{t} - v')\Big|_0^v, \tag{1.8.3}$$

where $v_\mathrm{t} \equiv mg/b$. Thus,

$$t = -\frac{m}{b}\big(\ln(v_\mathrm{t} - v) - \ln v_\mathrm{t}\big) = \frac{m}{b}\ln\left(\frac{v_\mathrm{t}}{v_\mathrm{t} - v}\right)$$

[5] F&C play havoc with −signs, using different conventions for the linear and quadratic cases in their §2.4, and making an error in their equation (2.4.13). My example is consistent with how I deal with −signs in FBDs throughout these lecture notes.

$$\Rightarrow \quad \frac{v_t}{v_t - v} = e^{bt/m} \quad \Rightarrow \quad v_t - v = v_t e^{-bt/m}$$

$$\Rightarrow \quad \boxed{v(t) = v_t\big(1 - e^{-t/\tau}\big),} \tag{1.8.4}$$

where $\tau \equiv m/b$, the so-called *characteristic time*.

- at $t = 0$, $v(0) = 0$, as required (m starts from rest)
- as $t \to \infty$, $v(t) \to v_t$, the *terminal velocity*.

Note that v_t can be found by setting $a_t = 0$ at terminal velocity:

$$F(v_t) = mg - bv_t = ma_t = 0 \quad \Rightarrow \quad v_t = \frac{mg}{b}.$$

Next, from Eq. (1.8.2) and $x_0 = 0$,

$$x = m\int_0^v \frac{v'dv'}{mg - bv'} = \frac{m}{b}\int_0^v \frac{v' - v_t + v_t}{v_t - v'}\,dv' = \tau\int_0^v \left(\frac{v_t}{v_t - v'} - 1\right)dv'$$

$$= v_t\tau\int_0^v \frac{dv'}{v_t - v'} - \tau\int_0^v dv' = -v_t\tau\big(\ln(v_t - v) - \ln v_t\big) - v\tau,$$

using integral from Eq. (1.8.3). Substitute Eq. (1.8.4) for v to find:

$$x = v_t\tau\big(\ln v_t - \ln(v_t e^{-t/\tau})\big) - v_t\tau\big(1 - e^{-t/\tau}\big)$$

$$= v_t\tau\big(\cancel{\ln v_t} - \cancel{\ln v_t} - \underbrace{\ln e^{-t/\tau}}_{-t/\tau} - 1 + e^{-t/\tau}\big)$$

$$\Rightarrow \quad \boxed{x(t) = v_t t - v_t\tau\big(1 - e^{-t/\tau}\big).}$$

- at $t = 0$, $x(0) = 0$, as required (we set $x_0 = 0$)
- as $t \to \infty$, $x(t) \to v_t t$, as expected when $v = v_t$ for most of time.

Exercise: Find $x(t)$ in a much easier way by noting that:

$$v(t) = \frac{dx(t)}{dt} \quad \Rightarrow \quad dx = v(t)dt \quad \Rightarrow \quad \int_0^x dx' = x = \int_0^t v(t')dt',$$

and use Eq. (1.8.4) for $v(t')$.

Tutorial 2.2

Problem 2 A baseball ($m = 0.145\,\mathrm{kg}$) is dropped from rest from the top of the CN tower ($h = 553\,\mathrm{m}$) and hits the ground at its *terminal velocity*, $v_\mathrm{t} = 42.5\,\mathrm{m\,s^{-1}}$ (where forces of gravity and air drag balance).

a) How much work is done by gravity?

b) How much work is done by air drag?

c) If there were no air drag, what would be the speed of the ball when it hits the ground?

Note: For rectilinear motion, work done by a force, F, acting in the same (+) or opposite (−) direction as the displacement between x_0 and x is:

$$W_F = \pm \int_{x_0}^{x} F\,dx'. \tag{1}$$

Problem 3 (FC 2.17)

a) If $F(x, v) = f(x)g(v)$, show that the differential equation of motion,

$$F(x, v) = m\ddot{x}, \tag{1}$$

can be solved for $x(t)$ by separation of variables and thus direct integration.

b) Can separation of variables be used to solve Eq. (1) if $F(v, t) = g(v)h(t)$? What if $F(x, t) = f(x)h(t)$?

Lesson 7

This lesson starts Part II on *Oscillations*. It includes:

1. the second of three lessons on solving ordinary differential equations (ODEs) where we:
   - distinguish between *linear* and *non-linear* ODEs,
   - introduce *second-order* ODEs, and
   - show that any second order ODE has two *linearly independent solutions*; and
2. an introduction to *Hooke's Law* which, combined with Newton's 2^{nd} law, yields a second order ODE which we,
   - solve for the simplest case (undamped, unforced),
   - reveal the *natural angular frequency of oscillation*, ω_0.

Part II: Oscillators

Reading assignment: Chapter 3, §3.1 – 3.4, 3.6

(*Show handout* `expansions.pdf`.)

2.1 Second lesson in solving ODEs

In the polynomial,

$$y(x) \;=\; a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \;=\; \sum_{i=0}^{n} a_i x^i,$$

where a_i are constants (no x-dependence),

- a_0 is the *constant* term,
- $a_1 x$ is the *linear* term ($y = a_1 x$ is a line!),
- $a_2 x^2$ is the *quadratic* term,
- $a_3 x^3$ is the *cubic* term, *etc.*

A polynomial ending $\left\{ \begin{array}{l} \text{at linear term} \\ \text{beyond linear term} \end{array} \right\}$ is said to be $\left\{ \begin{array}{l} \textit{linear} \\ \textit{non-linear} \end{array} \right\}$.

Since functions like e^x, $\ln(1+x)$, $\cos x$, $\sin x$ all have *power-law expansions*:

$$e^x \;=\; 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \;=\; \sum_{i=0}^{\infty} \frac{x^i}{i!} \tag{2.1.1}$$

$$\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \;=\; \sum_{i=1}^{\infty} \frac{(-1)^{i+1} x^i}{i} \tag{2.1.2}$$

$$\cos x \;=\; 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \;=\; \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}; \tag{2.1.3}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}, \tag{2.1.4}$$

they are also said to be non-linear.

ODEs (and PDEs) can be *linear* and *non-linear* as well.

A *general* 2[nd] order ODE has form:

$$\frac{d^2y}{dx^2} = y''(x) = f(y', y, x), \tag{2.1.5}$$

where f is *any* function of y', y, and x.

If $f(y', y, x)$ is a linear function of y' and y (but still may be non-linear in x), ODE is *linear*. Otherwise, ODE is *non-linear*.

Examples:

$$y''(x) = xe^{y'} + y\sin(y^2), \qquad \text{non-linear ODE;}$$

$$y''(x) = \frac{1}{y} + yy' + xy' + x^2, \qquad \text{non-linear ODE;}$$

$$y''(x) = -\frac{1}{x}y' + \left(\frac{n^2}{x^2} - 1\right)y, \qquad \text{linear ODE; Bessel's equation;}$$

$$\ddot{x}(t) = -\frac{c}{m}\dot{x} - \frac{k}{m}x, \qquad \text{linear ODE; damped oscillator.}$$

Most ODEs/PDEs in physics are linear, but by no means all!

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = f(x), \quad \text{non-linear PDE; Euler's equation (fluid dynamics).}$$

A *general* linear 2[nd] order ODE has form:

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = \begin{cases} 0, & \textit{homogeneous;} \\ d(x) \neq 0, & \textit{inhomogeneous,} \end{cases} \tag{2.1.6}$$

where a, b, c, d are known functions of x (perhaps constant).

Theorem 2.1. Principle of superposition. *If $y_1(x)$ and $y_2(x)$ both solve the linear homogeneous ODE Eq. (2.1.6), then so does a* linear combination*:*

$$y_3(x) \;=\; Ay_1(x) + By_2(x), \tag{2.1.7}$$

where A, B are any *constants.*

Proof. If we substitute Eq. (2.1.7) into LHS of Eq. (2.1.6) and get 0, then Eq. (2.1.7) solves the homogeneous Eq. (2.1.6). To that end,

$$\begin{aligned} ay_3'' + by_3' + cy_3 \;&=\; a(Ay_1'' + By_2'') + b(Ay_1' + By_2') + c(Ay_1 + By_2) \\ &=\; A(\underbrace{ay_1'' + by_1' + cy_1}_{0}) + B(\underbrace{ay_2'' + by_2' + cy_2}_{0}) \;=\; 0, \end{aligned}$$ □

Theorem 2.1 actually applies to any *linear* n^{th} order ODE.

Since A, B arbitrary, Eq. (2.1.7) represents an ∞ of solutions to Eq. (2.1.6). However,

Theorem 2.2. (stated without proof): *A* 2^{nd} *order ODE has two* linearly independent *solutions (one not a multiple of the other) from which all other solutions can be constructed [e.g., by Eq. (2.1.7)].*

In general, an n^{th} order ODE has n linearly independent solutions.

2.2 Hooke's law and simple harmonic oscillators (FC §3.1, 3.2)

Definition 2.1. *Harmonic oscillator*: Any system undergoing repetitive motion that exhibits:

- a regular *period* T (time for motion to repeat itself), and thus a regular *frequency* $f = 1/T$ (number of periods per second);

- a well-defined *amplitude*, A, where T may or may not depend upon A.

Definition 2.2. *Simple harmonic oscillator* (SHO): one whose motion can be described by a sine (or cosine) wave.

For an SHO, T is independent of A.

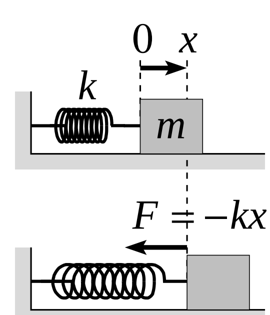
Demo: Three SHOs

Examples (*Show on-line handout* `harmonic.pdf`)

1. Harmonic oscillators
   - heart
   - Cepheid variables
   - exoplanet light curves
2. Simple harmonic oscillators
   - mass on a Hooke spring
   - low-amplitude pendulum
   - exoplanetary orbits about star

Robert Hooke (1635 – 1703) published *Hooke's Law* as a *Latin anagram* in 1676, then its solution in 1678:

as the extension, so the force



describing how a spring reacts to distortion.

Today, we write: $F = -kx$. Combined with Newton's second law (11 years later), we have:

$$F = -kx = m\ddot{x}, \tag{2.2.1}$$

a 2nd order ODE we'll solve "by inspection".

Rewriting Eq. (2.2.1) as:

$$\ddot{x} \;=\; -\frac{k}{m}x \;=\; -\omega_0^2 x, \tag{2.2.2}$$

where $\omega_0 \equiv \sqrt{\dfrac{k}{m}}$ has units s^{-1}, we ask:

What function, $x(t)$, has a second derivative equal to minus itself times a constant?

Dropping for the moment "times a constant", we can eliminate:

- an exponential, *e.g.* $x = e^t$, since $\dfrac{d^2x}{dt^2} = e^t = x$, not $-x$.
- a polynomial with finite number of terms, *e.g.* $x(t) = at^2 + bt + c$, since $\dot{x} = 2at + b \Rightarrow \ddot{x} = 2a \neq -x$.

On the other hand, if $x = \cos t$,

$$\frac{d^2x}{dt^2} \;=\; \frac{d}{dt}\frac{dx}{dt} \;=\; -\frac{d}{dt}\sin t \;=\; -\cos t \;=\; -x.$$

Similarly for $x = \sin t$.

To get the constant, ω_0^2, consider instead,

$$x_1(t) \;=\; \cos\omega_0 t \qquad \text{and} \qquad x_2(t) \;=\; \sin\omega_0 t,$$

as two *linearly independent* solutions to Eq. (2.2.2):

- $\sin\omega_0 t$ is not a multiple of $\cos\omega_0 t \;\Rightarrow\;$ linearly independent;
- Theorem 2.2 $\Rightarrow$ there must be two and only two.

Then,

$$\ddot{x}_1 \;=\; \frac{d^2x_1}{dt^2} \;=\; \frac{d}{dt}\frac{dx_1}{dt} \;=\; -\frac{d}{dt}(\omega_0\sin\omega_0 t) \;=\; -\omega_0^2\cos\omega_0 t \;=\; -\omega_0^2 x_1,$$

using chain rule twice. Bingo!

Exercise: Confirm same is true for $x_2(t)$.

Thus, most *general solution* to Eq. (2.2.2) is a *linear combination* of x_1, x_2,

$$x(t) \;=\; Ax_1(t) + Bx_2(t) \;=\; A\cos\omega_0 t + B\sin\omega_0 t, \tag{2.2.3}$$

(Theorem 2.1), where A, B are constants determined from initial conditions.

ω_0 is the *angular frequency* (rad s^{-1}), related to the *frequency*, f (number of oscillations per second), by,

$$\omega_0 \;=\; 2\pi f \;=\; \frac{2\pi}{T},$$

where T is the period of oscillation (Definition 2.1).

Exercise: Confirm Eq. (2.2.3) solves Eq. (2.2.2) by substituting Eq. (2.2.3) into LHS of Eq. (2.2.2), and showing it to be $-\omega_0^2 x(t)$.

Tutorial 3.1

Problem 1 Consider the following linear, second-order, homogeneous ODE:

$$\frac{d^2y}{dx^2} - 4y \;=\; 0. \tag{1}$$

a) Find "by inspection" two linearly independent solutions to equation (1).

b) From your two linearly independent solutions, write down the general solution.

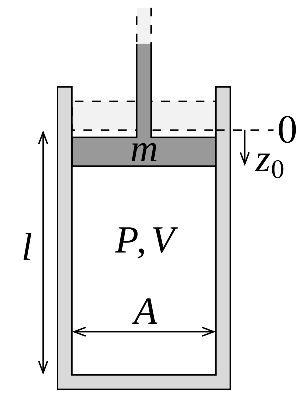
c) Show that when the boundary conditions $y(0) = 0$ and $y'(0) = 1$ are applied to your general solution in part b, you get:

$$y(x) \;=\; \tfrac{1}{2}\sinh 2x,$$

where $\sinh z \;\equiv\; \frac{1}{2}(e^z - e^{-z})$ is the *hyperbolic sine function*, which we'll meet in an upcoming class.

Problem 2 A piston of mass m and cross-sectional area A slides without friction and air leak in a cylinder of length l closed at one end. In the vertical position, it rests on top of an air column with equilibrium volume V and pressure P (the latter being the sum of the atmospheric pressure, P_{atm}, and the pressure exerted by the weight of the piston, mg/A).

From the equilibrium position, the piston is pushed into the cylinder by a distance $z_0 \ll l$, then released. The system oscillates rapidly enough that the gas may be considered *adiabatic*; heat neither enters nor leaves the gas as it repeatedly contracts and expands under the oscillation. For such a gas, the relationship between pressure and volume is given by the adiabatic gas law:



$$PV^{\gamma} = \kappa \quad \text{(constant)}, \tag{1}$$

where γ is the *ratio of specific heats* (5/3 for a monatomic gas, 7/5 for a diatomic gas, 4/3 for a polyatomic gas).

a) Make the approximation,

$$\delta P \approx \frac{dP}{dV}\,\delta V, \tag{2}$$

where $\delta P \ll P$ and $\delta V \ll V$, to show that for small displacements from equilibrium, z ($-z_0 \leq z \leq z_0$), the unbalanced force is given by,

$$F = A\delta P = -\frac{\gamma P A}{l}\,z. \tag{3}$$

b) Show how equation (3) represents a simple harmonic oscillator with a period given by,

$$T = 2\pi\sqrt{\frac{ml}{\gamma P A}}.$$

c) Find a numerical value for the period of oscillation for $m = 1.00\,\text{kg}$, $l = 1.00\,\text{m}$, $A = 0.125\,\text{m}^2$, and assuming a diatomic gas such as the atmosphere (to a darn good approximation). For P_{atm}, use $1.013 \times 10^5\,\text{N}\,\text{m}^{-2}$ and for g, use $9.81\,\text{m}\,\text{s}^{-2}$.

Lesson 8

This lesson continues our discussion on undamped, unforced simple harmonic oscillations (SHO) with:

1. three detailed examples;
2. noting the equivalence between SHO and circular motion; and
3. showing how to deal with the force of gravity in vertical oscillations.

Example 2.1. Suppose m is pulled (spring is stretched) to x_0 and released from rest at $t = 0$. Find $x(t)$.

Demo: Three SHOs

Solution: Setting $t = 0$ in Eq. (2.2.3),

$$x(0) \;=\; x_0 \;=\; A\cos 0 + B\sin 0 \;=\; A \quad\Rightarrow\quad A \;=\; x_0.$$

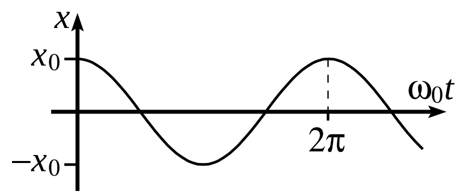
Next, on differentiating Eq. (2.2.3),

$$v(t) \;=\; \dot{x}(t) \;=\; -A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t \tag{2.2.4}$$

$$\Rightarrow\quad v(0) \;=\; 0 \;=\; -A\omega_0\sin 0 + B\omega_0\cos 0 \;=\; B\omega_0 \quad\Rightarrow\quad B \;=\; 0.$$

Thus, Eq. (2.2.3) becomes,

$$x(t) \;=\; x_0\cos\omega_0 t. \tag{2.2.5}$$



Example 2.2. At $t = 0$, m is at $x = 0$ and struck by a sharp blow imparting upon it an initial velocity v_0. Find $x(t)$.

Solution: Setting $t = 0$ in Eq. (2.2.3),

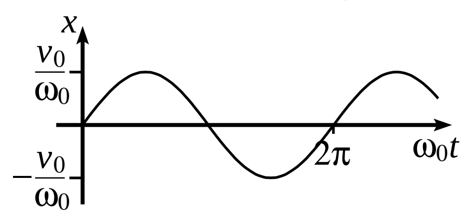
$$x(0) \;=\; 0 \;=\; A\cos 0 + B\sin 0 \;=\; A \quad\Rightarrow\quad A \;=\; 0.$$

Next, using Eq. (2.2.4),

$$v(0) \;=\; v_0 \;=\; -A\omega_0\sin 0 + B\omega_0\cos 0 \;=\; B\omega_0 \quad\Rightarrow\quad B \;=\; \frac{v_0}{\omega_0}.$$

Thus, Eq. (2.2.3) becomes,

$$x(t) \;=\; \frac{v_0}{\omega_0}\sin\omega_0 t.$$



Recap: Whenever we see an ODE of the form,

$$\boxed{\frac{d^2x}{dt^2} \;=\; \ddot{x} \;=\; -\omega_0^2 x \qquad \text{or} \qquad \frac{d^2y}{dx^2} \;=\; y'' \;=\; -\kappa^2 y \qquad \text{or}} \tag{2.2.6}$$

second derivative of ?? is minus a positive number times ??, we know we have an SHO whose solution we can write down right away.

If independent variable is time, t ($\ddot{x} = -\omega_0^2 x$):

- solution is $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$;
- A, B determined from *initial conditions*;
- *period* of oscillation $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega_0}$.

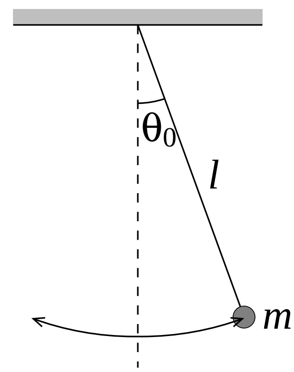
If independent variable is position, x ($y'' = -\kappa^2 y$):

- solution is $y(x) = A\cos\kappa x + B\sin\kappa x$, where $\kappa =$ *wave number*;
- A, B determined from *boundary conditions*;
- *wavelength* of oscillation $\lambda = \dfrac{2\pi}{\kappa}$.

General solution (with two constants set by two initial/boundary conditions) comes in three equivalent forms:

$$x(t) \;=\; A\cos\omega_0 t + B\sin\omega_0 t \;=\; x_0\cos(\omega_0 t - \phi_0) \;=\; ae^{i\omega_0 t} + be^{-i\omega_0 t}.$$

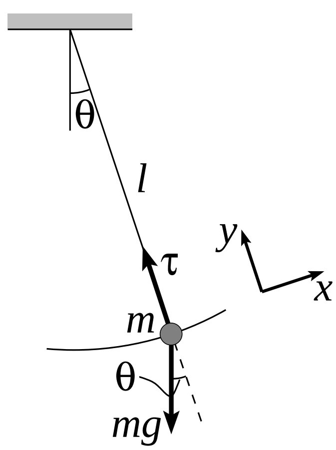
- *e.g.*, $x_0\cos(\omega_0 t - \phi_0) \;=\; \underbrace{x_0\cos\phi_0}_{A}\cos\omega_0 t + \underbrace{x_0\sin\phi_0}_{B}\sin\omega_0 t.$
- $x_0 =$ *amplitude* of oscillation
- $\phi_0 =$ *phase*, starting point of oscillator: $x(0) = x_0\cos\phi_0$.
- third form, $e^{\pm i\omega_0 t}$, examined later (Theorem 2.3).



Example 2.3. A *simple pendulum* (point mass, m, on a massless string, length l) is drawn a small angle θ_0 from vertical, and released from rest. Find its period of oscillation and equation of motion, $\theta(t)$.

Solution. Treat this as a dynamics problem.

Steps 1 & 2: Create FBD and break $\sum \vec{F} = m\vec{a}$ into components.



$$x/\ -mg\sin\theta\ =\ ma_x; \qquad y/\ \tau^{6} - mg\cos\theta\ =\ ma_y,$$

Step 3: constraints. Since m moves in circular path, we have from Eq. (1.2.3):

$$a_x\ =\ a_{\tan}\ =\ \ddot{\theta}\, l; \qquad a_y\ =\ a_{\rm cp}\ =\ \dot{\theta}^2 l,$$

Step 4: algebra. We care only about x-direction:

$$-\cancel{m}g\sin\theta\ =\ \cancel{m}l\ddot{\theta}.$$

For small angles, $\sin\theta \approx \theta$ (radians), and we have: $\ddot{\theta} \approx -\dfrac{g}{l}\theta$, a SHO, where,

$$\omega_0\ =\ \sqrt{\frac{g}{l}} \quad \Rightarrow \quad \boxed{T\ =\ \frac{2\pi}{\omega_0}\ =\ 2\pi\sqrt{\frac{l}{g}},} \tag{2.2.7}$$

is period of oscillation. For $\theta(t)$, general SHO solution is:

$$\theta(t)\ =\ A\cos\omega_0 t + B\sin\omega_0 t,$$

where initial conditions are applied to find A and B. First, find $\dot{\theta}$[7]:

$$\dot{\theta}(t)\ =\ -\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t,$$

[6] I need to use τ for tension because T is used later for period.

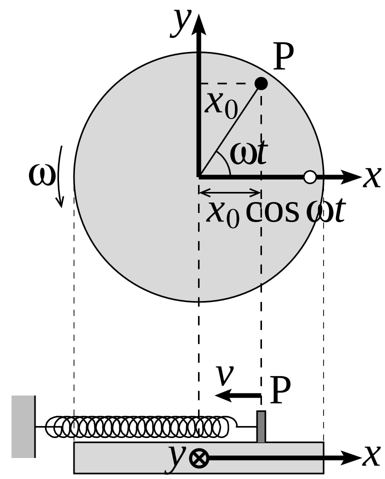
[7] There is great potential for confusion here on the use of the symbol ω. Elsewhere, ω is used for $\dot{\theta}$, the *angular velocity*, whereas for oscillators, F&C use ω for the *angular frequency*. To help alleviate this, I use ω_0 for the SHO frequency (though later, ω is used for the frequency of the driving force).

Then, at $t = 0$, $\theta(0) = \theta_0 = A$ and $\dot{\theta}(0) = 0 = \omega_0 B$,

$$\Rightarrow \quad \boxed{\theta(t) \;=\; \theta_0 \cos\omega_0 t \;=\; \theta_0 \cos\left(\sqrt{\frac{g}{l}}\,t\right),}$$

is equation of motion.

Simple harmonic and circular motions



A peg, P, moves clockwise on a turntable at constant angular speed ω and distance x_0 from centre. Let P be at $(x_0, 0)$ at $t = 0$ (on $+x$-axis).

At time t, P rotates by angle $\theta = \omega t$ and its x-coordinate is:

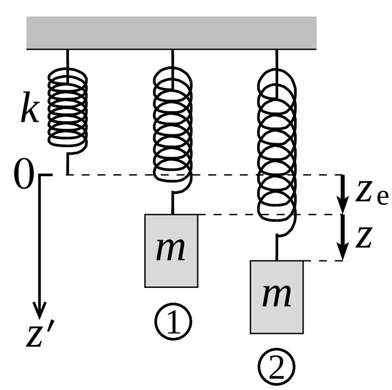
$$x(t) \;=\; x_0 \cos\omega t,$$

identical to Eq. (2.2.5) from Example 2.1 when $\omega = \omega_0$.

Thus, there is no difference between motion of P viewed "edge-on", and motion of mass, m, oscillating on a spring.

(*Show on-line handout* `SHO_circle.pdf` *where Youtube video link is.*)

Vertical oscillations

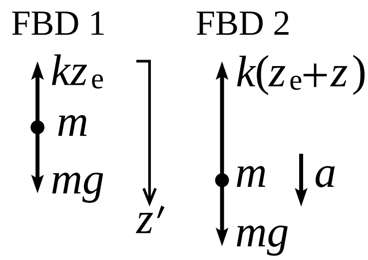


Hang an ideal (massless) spring vertically with no stretch (bottom of spring at $z' = 0$).

Attach mass, m, to end of spring; gently lower it to new equilibrium position ($z' = z_e$).

From FBD 1:

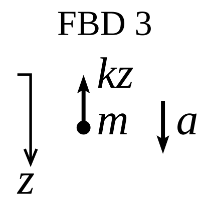
$$-\,kz_{\mathrm{e}} + mg \;=\; 0 \quad\Rightarrow\quad kz_{\mathrm{e}} \;=\; mg. \qquad (2.2.8)$$



Next, pull m down additional z_0, and release.
From FBD 2, when total stretch is $z' = z + z_{\mathrm{e}}$:

$$-k(z_{\mathrm{e}}+z)+mg \;=\; ma \;=\; m\frac{d^2}{dt^2}(z_{\mathrm{e}}+z) \;=\; m\ddot{z},$$

since z_{e} is constant. But $kz_{\mathrm{e}} = mg$ (Eq. 2.2.8), and thus,



$$-k(z_{\mathrm{e}}+z)+mg \;=\; -\cancel{kz_{\mathrm{e}}} - kz + \cancel{mg} \;=\; m\ddot{z}$$

$$\Rightarrow \quad -kz \;=\; m\ddot{z}, \qquad (2.2.9)$$

same equation of motion as horizontal spring; corresponds to FBD 3.

- z is displacement from new equilibrium position
- constant external force (*e.g.*, gravity) eliminated from equation of motion by setting $z = 0$ at new equilibrium position.

Tutorial 3.2

Problem 3 (FC 3.5) A particle undergoing simple harmonic motion has a velocity $\dot{x}_1$ when the displacement is x_1, and a velocity $\dot{x}_2$ when the displacement is x_2. In terms of the given quantities, find the:

a) angular frequency, ω_0;

b) period, T; and

c) amplitude of the motion, x_0.

Hint: It may be easier if you use the SHO solution form: $x(t) = x_0 \cos(\omega_0 t - \phi_0)$. Alternately, you could consider the mechanical energy of the oscillator.

Lesson 9

This lesson discusses the energetics of a simple harmonic oscillator (SHO) and begins the unit on *damped harmonic oscillation*. On the former, we:

1. show how to identify a simple harmonic oscillator from the form of its energy;
2. relate the maximum oscillator amplitude and speed to the energy of oscillation; and
3. apply what we've learned to a simple pendulum.

On the latter, we:

1. introduce how *damping* that forever reduces the oscillation amplitude is modelled;
2. write down the governing ODE; and
3. solve the ODE by *trial exponentials*.

2.3 Energy of an SHO (FC §3.3)

Since $F = -kx$ depends on position only, we can associate a potential energy with a spring. From Eq. (1.7.8) and since $k = m\omega_0^2$ (Eq. 2.2.2):

$$U(x) - U(0) = -\int_0^x F(x')\,dx' = \int_0^x kx'dx' = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2x^2.$$

If $U(0) = 0$ ($x = 0$ is equilibrium position), total energy of SHO is:

$$\boxed{E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2x^2 = \text{constant.}} \tag{2.3.1}$$

System is an SHO if E is quadratic in both position and "position-dot".

Since Eq. (2.3.1) was derived essentially by integrating Eq. (2.2.1; Hooke's Law), then:

- Eq. (2.3.1), (2.2.1) are equivalent, both equally identify an SHO;
- ratio of x^2 and $\dot{x}^2$ coefficients gives ω_0^2.

Note that at $x = x_0$, $\dot{x} = 0$. Thus,

$$E = \frac{1}{2}kx_0^2 \quad \Rightarrow \quad x_0 = \sqrt{\frac{2E}{k}}, \tag{2.3.2}$$

gives amplitude of oscillation in terms of total energy.

Example 2.4. Find $x(t)$ from Eq. (2.3.1).

Solution: From Eq. (1.7.9),

$$v(x) = \pm\sqrt{\frac{2}{m}(E - U(x))} = \pm\sqrt{\frac{2}{m}(E - \tfrac{1}{2}kx^2)}.$$

From Eq. (2.3.2), $E = \frac{1}{2}kx_0^2$ and thus,

$$v(x) = \pm\sqrt{\frac{k}{m}(x_0^2 - x^2)} = \pm\omega_0\sqrt{x_0^2 - x^2},$$

where $\pm$ means m could be moving in either direction.

For $x(t)$, use Eq. (1.7.10) with $t_0 = 0$ to get:

$$t = \int_0^x \frac{dx'}{v(x')} = \pm\frac{1}{\omega_0}\int_0^x \frac{dx'}{\sqrt{x_0^2 - x'^2}} = \mp\frac{1}{\omega_0}\underbrace{\cos^{-1}(x/x_0)}_{\text{integral tables}}.$$

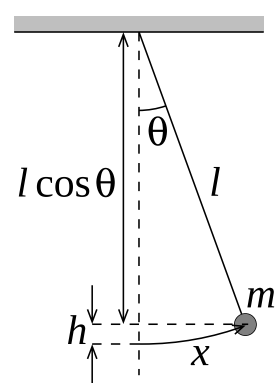
$$\Rightarrow \quad \boxed{x(t) = x_0\cos(\pm\omega_0 t) = x_0\cos\omega_0 t,}$$

identical to Eq. (2.2.5) found by solving 2nd order ODE $\ddot{x} = -\omega_0^2 x$ (Eq. 2.2.1).

Example 2.5. Find total energy for a simple pendulum.

Solution: Relative to bottom of swing, potential energy is,

$$U = mgh = mg(l - l\cos\theta) = mgl(1 - \cos\theta).$$



Now,

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots \approx 1 - \frac{\theta^2}{2},$$

for small θ. Thus,

$$U \approx mgl\left(1 - 1 + \frac{\theta^2}{2}\right) = \frac{mgl\theta^2}{2} = \frac{mg}{2l}x^2,$$

since $x = \theta l$ (arc length, Eq. 1.2.4).

$U(x)$ quadratic in $x \Rightarrow$ simple pendulum is an SHO (for small θ).

Thus, total energy is:

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{mg}{2l}x^2 \quad \Rightarrow \quad \omega_0^2 = \frac{mg/2l}{m/2} = \frac{g}{l},$$

identical to Eq. (2.2.7).

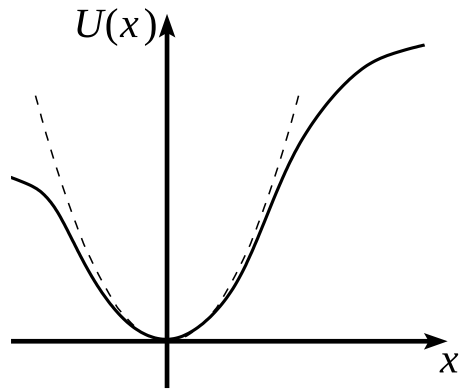
Every system is an SHO for small displacements from local minima in $U(x)$:

Do a *Maclaurin* expansion on $U(x)$,

$$U(x) \;=\; U(0) + U'(0)x + \frac{U''(0)}{2}x^2 + \frac{U'''(0)}{3!}x^3 + \cdots .$$

Let $U(0) = 0$. Since $x = 0$ is a local minimum, $U'(0) = 0$. Thus, for $x \ll 1$,

$$U(x) \;\approx\; \frac{U''(0)}{2}x^2,$$



a quadratic in x! Thus, m exhibits SH motion about $x = 0$ with frequency,

$$\boxed{\omega_0^2 \;=\; \frac{U''(0)}{m}.} \qquad (2.3.3)$$

e.g., for a Hooke's spring,

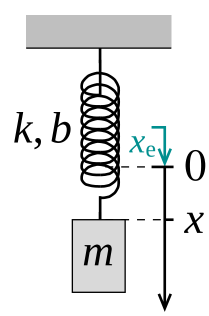
$$U(x) \;=\; \frac{1}{2}kx^2 \quad\Rightarrow\quad U''(x) \;=\; k \quad\Rightarrow\quad U''(0) \;=\; k \quad\Rightarrow\quad \omega_0^2 \;=\; \frac{k}{m},$$

as expected.

End material for midterm 1.

2.4 Damped harmonic motion (FC §3.4)

Demo: damped, forced SHO



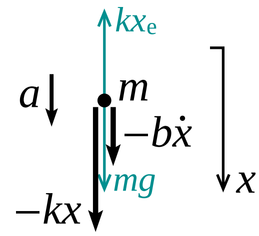
Model *damping* (frictional forces internal to spring) as we did linear air drag (§1.8)[8]:

$$F_{\text{damp}} \;=\; -b\dot{x},$$

$-$sign $\Rightarrow$ opposite to direction of motion.

[8]I use b rather than c as F&C do to be consistent with air drag in §1.8, and because at times we'll want c for the speed of light.

Set $x = 0$ to equilibrium point (x_e) to eliminate mg and kx_e from FBD and equation of motion (Eq. 2.2.9).



Not knowing direction of $-kx$ and $-b\dot{x}$, point them in $+x$-direction, label them with their minus signs on FBD.

From Newton's 2nd law, in x-direction:

$$\sum F = ma \quad \Rightarrow \quad (-kx) + (-b\dot{x}) = m\ddot{x}$$

$$\Rightarrow \quad \boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,} \tag{2.4.1}$$

where:

$$\gamma = \frac{b}{2m} = \textit{damping coefficient};$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \text{frequency of undamped oscillator.}$$

Eq. (2.4.1) is a *homogeneous 2nd order, linear ODE with constant coefficients*, which we'll solve by *trial exponentials* (mentioned in §1.6)[9].

That is, we'll *assume* a solution of the form $x(t) = e^{rt}$, and see what happens!

But first, *why on Earth would you guess that?!?*

Here's the thinking:

- Solving Eq. (2.4.1) by "inspection" is futile: omitting constants, what function's 2nd derivative is minus its 1st derivative minus itself? No way.

- Solving *any* integral or ODE requires "inspiration". Here, if number of "dots" were "powers" ($\ddot{x} \to x^2$, $\dot{x} \to x^1$, $x \to x^0$), Eq. (2.4.1) becomes:

$$x^2 + 2\gamma x + \omega_0^2 = 0,$$

and *that* we could solve for x! So how do we replace dots with powers?

[9]Text solves this equation by treating it as an operator equation. Students are already new to ODEs; introducing operators now is untenable.

- The fact that Eq. (2.4.1) has constant coefficients and is homogeneous (0 on RHS) is key. If time dependence in $\ddot{x}$, $\dot{x}$, x were the same, they'd cancel and we'd be left with an equation of constants.

- So what function is the same after differentiation? An exponential!

Substituting $x = e^{rt}$ into Eq. (2.4.1), we get:

$$r^2 \cancel{e^{rt}} + 2\gamma r \cancel{e^{rt}} + \omega_0^2 \cancel{e^{rt}} = 0 \quad \Rightarrow \quad r^2 + 2\gamma r + \omega_0^2 = 0,$$

which we solve for r using quadratic formula. Thus,

$$r = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm q,$$

where $q \equiv \sqrt{\gamma^2 - \omega_0^2}$.

For $q \neq 0$ ($\gamma \neq \omega_0$), these give two independent solutions to Eq. (2.4.1):

$$x_1(t) = e^{(-\gamma+q)t} = e^{-\gamma t} e^{qt}; \qquad x_2(t) = e^{(-\gamma-q)t} = e^{-\gamma t} e^{-qt}.$$

and general solution is (Theorems 2.1, 2.2; Eq. 2.1.7),

$$x(t) = Ax_1 + Bx_2 = e^{-\gamma t}\left(Ae^{qt} + Be^{-qt}\right) \tag{2.4.2}$$

$$\Rightarrow \quad \dot{x}(t) = -\gamma e^{-\gamma t}\left(Ae^{qt} + Be^{-qt}\right) + e^{-\gamma t} q\left(Ae^{qt} - Be^{-qt}\right),$$

where A, B are set from initial conditions. Suppose at $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = 0$. Then,

$$x(0) = A + B = x_0; \qquad \dot{x}(0) = -\gamma \underbrace{(A+B)}_{x_0} + q(A-B) = 0.$$

Solve for A and B: $\quad A + B = x_0; \qquad A - B = \dfrac{\gamma}{q} x_0$

$$\text{add:} \quad 2A = x_0 + \frac{\gamma}{q} x_0 \quad \Rightarrow \quad A = \frac{x_0}{2}\left(1 + \frac{\gamma}{q}\right);$$

$$\text{subtract:} \quad 2B = x_0 - \frac{\gamma}{q} x_0 \quad \Rightarrow \quad B = \frac{x_0}{2}\left(1 - \frac{\gamma}{q}\right)$$

$$
\begin{aligned}
\Rightarrow \quad x(t) &= \frac{x_0 e^{-\gamma t}}{2}\left[\left(1+\frac{\gamma}{q}\right)e^{qt} - \left(1-\frac{\gamma}{q}\right)e^{-qt}\right] \\
&= x_0 e^{-\gamma t}\left(\frac{e^{qt}+e^{-qt}}{2} + \frac{\gamma}{q}\frac{e^{qt}-e^{-qt}}{2}\right), \quad q \neq 0. \qquad (2.4.3)
\end{aligned}
$$

For $q = 0$, take limit as $q \to 0$ of Eq. (2.4.3). Thus,

$$
\begin{aligned}
\lim_{q\to 0} x(t) &= x_0 e^{-\gamma t}\left(1 + \frac{\gamma}{2}\lim_{q\to 0}\frac{e^{qt}-e^{-qt}}{q}\right) \quad \text{(use l'Hôpital)} \\
&= x_0 e^{-\gamma t}\left(1 + \frac{\gamma}{2}\lim_{q\to 0}\frac{te^{qt}+te^{-qt}}{1}\right) = x_0 e^{-\gamma t}(1+\gamma t).
\end{aligned}
$$

$$
\Rightarrow \quad x(t) = x_0 e^{-\gamma t}(1+\gamma t), \quad q = 0. \qquad (2.4.4)
$$

Tutorial 4.1

Problem 1

a) Show that Eq. (2.2.1), $\ddot{x} = -\omega_0^2 x$, can be derived from Eq. (2.3.1), $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 = \text{constant}$. *Hint*: Try differentiating the latter with respect to t.

b) In terms of E, what is the maximum speed of a SHO?

c) Use Eq. (2.3.3) to find the frequency of a simple pendulum.

Lesson 10

This lesson continues our discussion on *damped harmonic oscillation* where we:

1. introduce *Euler's formula* and *hyperbolic trig functions*; and
2. examine the four cases of damped harmonic oscillations,
    - *overdamped*,
    - *critically damped*,
    - *underdamped*,
    - *no damping*,

all as special cases of the general solution to the ODE derived in the previous lesson.

Aside: Euler's formula and hyperbolic trig functions

Theorem 2.3. *Euler's formula:*

$$\boxed{e^{i\theta} \;=\; \cos\theta + i\sin\theta,} \tag{2.4.5}$$

where $i = \sqrt{-1}$. *Who knew!*

Proof. Consider the Maclaurin expansion for $e^{i\theta}$ Eq. (2.1.1):

$$\begin{aligned}
e^{i\theta} \;=\; \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} - \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\
&= \underbrace{1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots}_{\cos\theta\ \text{(Eq. 2.1.3)}} + i\Bigg(\underbrace{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots}_{\sin\theta\ \text{(Eq. 2.1.4)}}\Bigg) \\
&= \cos\theta + i\sin\theta,
\end{aligned}$$

where we used $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = (i^2)^2 = 1$, and $i^5 = i^4 i = i$. □

Thus, we also have:

$$e^{-i\theta} \;=\; e^{i(-\theta)} \;=\; \cos(-\theta) + i\sin(-\theta) \;=\; \cos\theta - i\sin\theta. \tag{2.4.6}$$

Exercise: Show from Eq. (2.4.5), (2.4.6) that:

$$\cos\theta \;=\; \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta \;=\; \frac{e^{i\theta} - e^{-i\theta}}{2i}, \tag{2.4.7}$$

two exceedingly useful identities.

Hyperbolic trig functions, cosh and sinh, are defined like Eq. (2.4.7) except with real exponential arguments:

$$\cosh\alpha \;=\; \frac{e^{\alpha}+e^{-\alpha}}{2} \qquad \text{and} \qquad \sinh\alpha \;=\; \frac{e^{\alpha}-e^{-\alpha}}{2}. \tag{2.4.8}$$

Ordinary trig functions are sometimes referred to as *elliptical trig functions*.

Now, let $\alpha = i\theta$. Then Eq. (2.4.8) becomes:

$$\cosh i\theta \;=\; \frac{e^{i\theta}+e^{-i\theta}}{2} \;=\; \cos\theta; \qquad \sinh i\theta \;=\; \frac{e^{i\theta}-e^{-i\theta}}{2} \;=\; i\sin\theta.$$

Thus, $\theta = -i\alpha$ and Eq. (2.4.7) becomes:

$$\cos(-i\alpha) \;=\; \cos i\alpha \;=\; \frac{e^{i(-i\alpha)}+e^{-i(-i\alpha)}}{2} \;=\; \frac{e^{\alpha}+e^{-\alpha}}{2} \;=\; \cosh\alpha;$$

$$\sin(-i\alpha) \;=\; -\sin i\alpha \;=\; \frac{e^{i(-i\alpha)}-e^{-i(-i\alpha)}}{2i} \;=\; \frac{e^{\alpha}-e^{-\alpha}}{2i} \;=\; -i\sinh\alpha.$$

$$\Rightarrow \boxed{\begin{array}{ll} \cosh i\theta \;=\; \cos\theta, & \sinh i\theta \;=\; i\sin\theta; \\ \cos i\alpha \;=\; \cosh\alpha, & \sin i\alpha \;=\; i\sinh\alpha. \end{array}} \tag{2.4.9}$$

Hyperbolic trig identities:

$$\cosh^2\alpha - \sinh^2\alpha \;=\; 1; \qquad \tanh\alpha \;=\; \frac{\sinh\alpha}{\cosh\alpha};$$

$$\cosh(\alpha+\beta) \;=\; \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta; \qquad \frac{d\cosh\alpha}{d\alpha} \;=\; \sinh\alpha;$$

$$\sinh(\alpha+\beta) \;=\; \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta; \qquad \frac{d\sinh\alpha}{d\alpha} \;=\; \cosh\alpha.$$

Back to Eq. (2.4.3), (2.4.4) ...

For $q = \sqrt{\gamma^2 - \omega_0^2} = 0$, $\gamma = \omega_0$ and oscillator is said to be *critically damped*.

For $q \neq 0$:

- $\gamma > \omega_0 \;\Rightarrow\; q \in \mathbb{R}$; oscillator is *over-damped*;

- $\gamma < \omega_0 \;\Rightarrow\; q \in \mathbb{I}$; oscillator is *under-damped*.

Case 1: Overdamped system ($\gamma > \omega_0$; $q \in \mathbb{R}$).

Using identities (2.4.8), Eq. (2.4.3) becomes:

$$x(t) \;=\; x_0 e^{-\gamma t}\left(\frac{e^{qt}+e^{-qt}}{2} + \frac{\gamma}{q}\frac{e^{qt}-e^{-qt}}{2}\right) \;=\; x_0 e^{-\gamma t}\left(\cosh qt + \frac{\gamma}{q}\sinh qt\right), \tag{2.4.10}$$

purely exponential in character (no oscillations).

As $t \to \infty$, $\cosh qt \to \sinh qt \to \frac{1}{2}e^{qt}$, and displacement of oscillator,

$$x(t) \;\to\; x_0 e^{-\gamma t}\left(\frac{qe^{qt}+\gamma e^{qt}}{2q}\right) \;=\; x_0 \frac{q+\gamma}{2q}\, e^{-(\gamma-q)t},$$

decays exponentially $\left(\gamma - q = \gamma - \sqrt{\gamma^2-\omega_0^2} > 0\right)$.

Define "e-folding time", $\tau_{\rm od}$ = time for amplitude to fall by factor e. Thus,

$$x(t+\tau_{\rm od}) \;=\; x(t)e^{-1} \;=\; x_0\frac{q+\gamma}{2q}e^{-(\gamma-q)(t+\tau_{\rm od})} \;=\; \underbrace{x_0\frac{q+\gamma}{2q}e^{-(\gamma-q)t}}_{x(t)}\;\underbrace{e^{-(\gamma-q)\tau_{\rm od}}}_{e^{-1}}$$

$$\Rightarrow \quad (\gamma-q)\tau_{\rm od} \;=\; 1 \quad \Rightarrow \quad \tau_{\rm od} \;=\; \frac{1}{\gamma-q} \;=\; \frac{1}{\gamma-\sqrt{\gamma^2-\omega_0^2}}.$$

Case 2: Critically damped system ($\gamma = \omega_0$; $q = 0$).

From Eq. (2.4.4),

$$x(t) \;=\; x_0 e^{-\gamma t}(1+\gamma t),$$

also exponential in character (no oscillations), with e-folding time (as $t \to \infty$),

$$\tau_{\rm cd} \;=\; \frac{1}{\gamma} \;=\; \frac{1}{\omega_0}.$$

Since $\tau_{\rm cd} < \tau_{\rm od}$, critically damped system decays faster than overdamped.

Case 3: Underdamped system ($\gamma < \omega_0$; $q \in \mathbb{I}$).

In this case, $q = \sqrt{\gamma^2 - \omega_0^2} \in \mathbb{I}$. Thus, define $\omega_{\rm d} = iq$:

$$\omega_{\rm d} = i\sqrt{\gamma^2 - \omega_0^2} = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \in \mathbb{R}.$$

Substitute $q = \omega_{\rm d}/i = -i\omega_{\rm d}$ into Eq. (2.4.10) and, with identities (2.4.9):

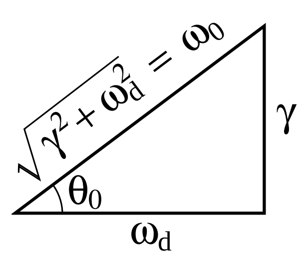
$$\begin{aligned} x(t) &= x_0 e^{-\gamma t}\left(\cosh(-i\omega_{\rm d})t + \frac{\gamma}{-i\omega_{\rm d}}\sinh(-i\omega_{\rm d})t\right) \\ &= x_0 e^{-\gamma t}\left(\cosh i\omega_{\rm d}t + \frac{\gamma}{i\omega_{\rm d}}\sinh i\omega_{\rm d}t\right) \\ &= x_0 e^{-\gamma t}\left(\cos\omega_{\rm d}t + \frac{\gamma}{\omega_{\rm d}}\sin\omega_{\rm d}t\right), \end{aligned} \tag{2.4.11}$$

Underdamped system is oscillatory with exponentially decaying amplitude:

- amplitude $x_0 e^{-\gamma t}$ decays with e-folding time $\frac{1}{\gamma}$, same as critical damping;
- $\omega_{\rm d}$ is oscillation frequency of damped system.

Note that $\omega_{\rm d} < \omega_0 \Rightarrow$ damping slows oscillation frequency (increases period).

Consider right triangle with γ, $\omega_{\rm d}$ on short sides with γ opposite to "phase angle" θ_0. Then,



$$\gamma^2 + \omega_{\rm d}^2 = \cancel{\gamma^2} + \omega_0^2 - \cancel{\gamma^2} = \omega_0^2,$$

$\Rightarrow$ ω_0 (undamped oscillator frequency) on hypotenuse.

$$\Rightarrow \quad \tan\theta_0 = \frac{\gamma}{\omega_{\rm d}}; \quad \cos\theta_0 = \frac{\omega_{\rm d}}{\omega_0}; \quad \sin\theta_0 = \frac{\gamma}{\omega_0}, \tag{2.4.12}$$

and Eq. (2.4.11) can be written as,

Demo: underdamped SHO

$$
\begin{aligned}
x(t) &= x_0 e^{-\gamma t}\left(\cos\omega_\mathrm{d} t + \tan\theta_0 \sin\omega_\mathrm{d} t\right) \\
&= \frac{x_0 e^{-\gamma t}}{\cos\theta_0}\big(\cos\omega_\mathrm{d} t\cos\theta_0 + \sin\omega_\mathrm{d} t\sin\theta_0\big) \\
\Rightarrow \quad x(t) &= x_0 e^{-\gamma t}\frac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \theta_0). \qquad (2.4.13)
\end{aligned}
$$

Case 4: Undamped system ($\gamma = 0$).

For $\gamma = 0$, $\theta_0 = 0$ and $\omega_\mathrm{d} = \omega_0$, and Eq. (2.4.13) becomes:

$$x(t) = x_0 \cos\omega_0 t,$$

identical to Eq. (2.2.5).

Recap: Governing ODE for damped harmonic oscillator,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0,$$

has solution for $x(0) = x_0$ and $\dot{x}(0) = 0$:

$$
x(t) = x_0 \begin{cases}
e^{-\gamma t}\left(\cosh qt + \dfrac{\gamma}{q}\sinh qt\right), & \gamma > \omega_0 \text{ (overdamped)}; \\
e^{-\gamma t}(1+\gamma t), & \gamma = \omega_0 \text{ (critically damped)}; \\
e^{-\gamma t}\dfrac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \theta_0), & \gamma < \omega_0 \text{ (underdamped)}; \\
\cos\omega_0 t, & \gamma = 0 \text{ (undamped)},
\end{cases} \qquad (2.4.14)
$$

where:

$$
\begin{aligned}
x_0 &= \text{initial displacement of } m \text{ (stretch of spring)}; \\
\gamma &= \frac{b}{2m} = \text{damping coefficient};
\end{aligned}
$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \text{oscillation frequency of undamped system;}$$

$$q = \sqrt{\gamma^2 - \omega_0^2};$$

$$\omega_\mathrm{d} = iq = \sqrt{\omega_0^2 - \gamma^2} = \text{oscillation frequency of underdamped system;}$$

$$\theta_0 = \sin^{-1}\frac{\gamma}{\omega_0} = \text{phase lag of underdamped system.}$$

(*Show on-line handout* `damped.pdf`.)

Example 2.6. (Example 3.4.1 from text) A car suspension is critically damped with undamped oscillation period $T_0 = \dfrac{2\pi}{\omega_0} = 1\,\mathrm{s}$. If system is displaced x_0 at $t = 0$ and released from rest, find displacement, x, after 1 s.

Solution: From initial conditions, use second case of Eq. (2.4.14):

$$x(t) = x_0 e^{-\gamma t}(1 + \gamma t) = x_0 e^{-\omega_0 t}(1 + \omega_0 t),$$

since $\gamma = \omega_0 = \dfrac{2\pi}{T_0} = 2\pi\,\mathrm{rad\,s^{-1}}$ for critical damping. Thus, at $t = 1\,\mathrm{s}$,

$$x(1) = x_0 e^{-2\pi}(1 + 2\pi) \sim 0.0136 x_0.$$

Tutorial 4.2

Problem 2 (FC 3.9) Show that the ratio of two successive maxima in the displacement of an underdamped harmonic oscillator is constant.

Problem 3 An underdamped harmonic oscillator consists of a spring with spring constant $k = 10.0\,\mathrm{N\,m^{-1}}$ and a mass $m = 0.100\,\mathrm{kg}$. The mass is displaced from equilibrium by 4.00 cm and it is observed that after oscillating for 10 s, an integral number of oscillations have occurred where the amplitude is now just 2.00 cm.

a) What is the damping coefficient, γ?

b) What is the natural oscillation frequency, ω_d, and thus the period, T_d, of the damped oscillator?

Lesson 11

This lesson concludes our discussion on damped harmonic oscillations, and finishes with the third of three lessons in solving ODEs. Subtopics include:

1. the "*LRC circuit*", an example from electricity and magnetism of a damped harmonic oscillator;
2. the *quality factor* Q_{d}, a measure of how damped a system is;
3. finding the general solution to an *inhomogeneous* second order ODE with constant coefficients, and identifying the *homogeneous* and *particular* portions of that solution.

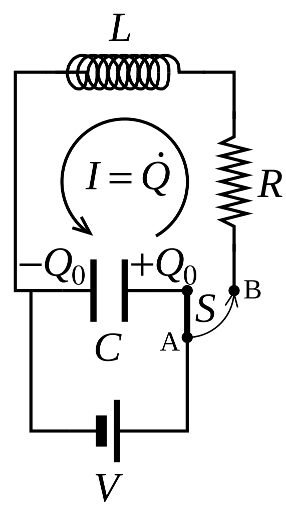
Example 2.7. The "LRC circuit"

Demo: LRC circuit

Consider an "LRC" circuit with inductance L, resistance R, capacitance C.

At $t < 0$, switch S is at A where DC supply, V, maintains charge $\pm Q_0$ on two capacitor plates.

At $t = 0$, S is moved to B (which disconnects V). Find current, $I(t)$, in LRC circuit for $t \geq 0$.



Solution: "Voltage drop" across each element is:

$$V_L = L\dot{I} = L\ddot{Q}; \qquad V_R = RI = R\dot{Q}; \qquad V_C = \frac{Q}{C}.$$

By Kirchhoff's first rule, $\sum V = 0$ in a closed loop. Thus, for $t \geq 0$,

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = 0 \quad \Rightarrow \quad \ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{LC}Q = 0,$$

same form as Eq. (2.4.1), with:

$$x(t) = Q(t); \qquad \gamma = \frac{R}{2L}; \qquad \omega_0^2 = \frac{1}{LC}.$$

Further, at $t = 0$, $Q = Q_0$ and $I = \dot{Q} = 0$. Thus, we have from Eq. (2.4.14):

$$Q(t) = Q_0 \begin{cases} e^{-\gamma t}\left(\cosh qt + \dfrac{\gamma}{q}\sinh qt\right), & \gamma > \omega_0 \text{ (overdamped)}; \\ e^{-\gamma t}(1 + \gamma t), & \gamma = \omega_0 \text{ (critically damped)}; \\ e^{-\gamma t}\dfrac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \theta_0), & \gamma < \omega_0 \text{ (underdamped)}; \\ \cos\omega_0 t, & \gamma = 0 \text{ (undamped)}, \end{cases}$$

where,

$$
\begin{aligned}
q &= \sqrt{\gamma^2 - \omega_0^2} = \frac{1}{L}\sqrt{\frac{R^2}{4} - \frac{L}{C}};\\
\omega_{\mathrm{d}} &= \sqrt{\omega_0^2 - \gamma^2} = \frac{1}{L}\sqrt{\frac{L}{C} - \frac{R^2}{4}};\\
\sin\theta_0 &= \frac{\gamma}{\omega_0} = \frac{R}{2}\sqrt{\frac{C}{L}}.
\end{aligned}
$$

How's your algebra? Finish example by finding $I(t) = \dot{Q}(t)$.

Quality factor

$$
\begin{aligned}
E &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2\\
\Rightarrow \quad \dot{E} &= m\dot{x}\ddot{x} + kx\dot{x} = \underbrace{(m\ddot{x} + kx)}_{-b\dot{x}\ \text{(Eq. 2.4.1)}}\dot{x} = -b\dot{x}^2 \le 0,
\end{aligned}
$$

damping force $\times$ speed (power). Thus, for $b > 0$ ($\gamma > 0$), E decays in time.

For weak damping, $\gamma \ll \omega_0$ and third case of Eq. (2.4.14) applies. Let $\phi = \omega_{\mathrm{d}}t - \theta_0$. Then,

$$
x(t) = x_0 e^{-\gamma t}\frac{\omega_0}{\omega_{\mathrm{d}}}\cos\phi \quad \Rightarrow \quad \dot{x}(t) = -x_0\gamma e^{-\gamma t}\frac{\omega_0}{\omega_{\mathrm{d}}}\cos\phi - x_0 e^{-\gamma t}\omega_{\mathrm{d}}\sin\phi.
$$

Since $\gamma \ll \omega_0 \sim \omega_{\mathrm{d}}$, first term in $\dot{x}(t)$ can be dropped. Thus,

$$
E(t) = \tfrac{1}{2}k\left(\frac{\dot{x}^2}{\omega_0^2} + x^2\right) \approx \underbrace{\tfrac{1}{2}kx_0^2}_{E_0}e^{-2\gamma t}\big(\underbrace{\sin^2\phi + \cos^2\phi}_{1}\big) = E_0 e^{-2\gamma t},
$$

and total energy falls from initial energy E_0 with "e-folding" time,

$$
\tau_E = \frac{1}{2\gamma}. \tag{2.4.15}
$$

Now, define *quality factor*, Q_{d}, to be:

$$
Q_{\mathrm{d}} = 2\pi\frac{E}{|\Delta E|},
$$

where ΔE is energy loss over a period $T_\mathrm{d} = \dfrac{2\pi}{\omega_\mathrm{d}}$. Then,

$$\Delta E = E_0 e^{-2\gamma(t+T_\mathrm{d})} - E_0 e^{-2\gamma t} = \underbrace{E_0 e^{-2\gamma t}}_{E} \big(\underbrace{e^{-2\gamma T_\mathrm{d}} - 1}_{<0}\big)$$

$$\Rightarrow \quad Q_\mathrm{d} = \frac{2\pi}{1 - e^{-2\gamma T_\mathrm{d}}} = \frac{2\pi}{\cancel{1} - \big(\cancel{1} - 2\gamma T_\mathrm{d} + \underbrace{\tfrac{1}{2}4\gamma^2 T_\mathrm{d}^2 + \cdots}_{\to 0}\big)} \approx \frac{2\pi}{2\gamma T_\mathrm{d}},$$

since, for $\gamma \ll \omega_0 \sim \omega_\mathrm{d} \Rightarrow \gamma T_\mathrm{d} = 2\pi \dfrac{\gamma}{\omega_\mathrm{d}} \ll 1$. Thus, with $2\gamma = 1/\tau_E$,

$$\begin{aligned} Q_\mathrm{d} \approx 2\pi \frac{\tau_E}{T_\mathrm{d}} &= 2\pi \times \text{number of periods for } E \text{ to decrease by } 1/e \\ &= \omega_\mathrm{d}\tau_E = \frac{\omega_\mathrm{d}}{2\gamma} \approx \frac{\omega_0}{2\gamma}. \end{aligned} \tag{2.4.16}$$

oscillator	Q_d
piano string	3×10^3
crystal in digital watch	10^4
neutron star	10^{12}

Example 2.8. For the LRC circuit,

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad 2\gamma = \frac{R}{L} \quad \Rightarrow \quad Q_\mathrm{d} \approx \frac{\omega_0}{2\gamma} = \frac{1}{\sqrt{LC}}\frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

Example 2.9. (Example 3.4.4 from text) A mass $m = 0.1\,\mathrm{kg}$ on a spring ($k = 0.4\,\mathrm{N\,m^{-1}}$; $\gamma = 0.01\,\mathrm{s^{-1}}$) is displaced x_0 and released from rest[10].

a) How many oscillations has m undergone when amplitude is $x_0/2$?

b) Find Q_d.

[10] These numbers roughly correspond to a sphere oscillating on an ideal spring immersed in water, with the water viscosity serving as the damping force.

Solution: Since $\gamma = 0.01 \ll \omega_0 \ = \ 2.0$, use third case of Eq. (2.4.14):

$$x(t) \ = \ \underbrace{x_0 e^{-\gamma t}}_{A(t)} \frac{\omega_0}{\omega_\mathrm{d}} \cos(\omega_\mathrm{d} t - \theta_0),$$

where $A(t)$ = amplitude of oscillation[11] with e-folding time,

$$\tau_A \ = \ \frac{1}{\gamma} \ = \ 2\tau_E,$$

from Eq. (2.4.15). Thus, A falls off at half the rate energy does. Continuing,

$$A(t_{1/2}) \ = \ \frac{x_0}{2} \ = \ x_0 e^{-\gamma t_{1/2}} \quad \Rightarrow \quad e^{-\gamma t_{1/2}} \ = \ \frac{1}{2} \quad \Rightarrow \quad t_{1/2} \ = \ \frac{\ln 2}{\gamma},$$

is time for A to halve. But time for one oscillation is,

$$T_\mathrm{d} \ = \ \frac{2\pi}{\omega_\mathrm{d}} \quad \Rightarrow \quad n \ = \ \frac{t_{1/2}}{T_\mathrm{d}} \ = \ \frac{\omega_\mathrm{d} \ln 2}{2\pi\gamma} \ = \ \frac{Q_\mathrm{d} \ln 2}{\pi},$$

using Eq. (2.4.16), is number of oscillations in time $t_{1/2}$. With numbers,

$$\omega_\mathrm{d} \ = \ \sqrt{\omega_0^2 - \gamma^2} \ = \ \sqrt{\frac{k}{m} - \gamma^2} \ \sim \ 2.00$$

$$\Rightarrow \quad n \ \sim \ \frac{(2.00) \ln 2}{2\pi(0.01)} \ \sim \ 22.1 \qquad \text{and} \qquad Q_\mathrm{d} \ = \ \frac{\pi n}{\ln 2} \ \sim \ 100.$$

Note that Q_d scales with n.

2.5 Third (and final) lesson in solving ODEs

Consider *inhomogeneous, linear* 2^{nd} *order ODE*:

$$a y''(x) + b y'(x) + c y(x) \ = \ d(x), \tag{2.5.1}$$

where a, b, c are constant.

Let $y_\mathrm{p}(x)$ be any function that solves Eq. (2.5.1):

[11] Should note here that the amplitude is *not* $x_0 e^{-\gamma t} \omega_0/\omega_\mathrm{d}$ since, at $t = 0$, $\cos(\omega_0 t - \theta_0) = \cos\theta_0 = \omega_\mathrm{d}/\omega_0$ (Eq. 2.4.12), cancelling out the factor $\omega_0/\omega_\mathrm{d}$.

- $\exists\ \infty$ of solutions to any ODE, $y_{\rm p}(x)$ is just one that solves Eq. (2.5.1);

- $y_{\rm p}(x)$ is not necessarily the general solution to Eq. (2.5.1);

- $y_{\rm p}(x)$ is known as the *particular solution.*

Further, let $y_{\rm h}(x)$ be *general solution* to *homogeneous* version of Eq. (2.5.1):

$$a y_{\rm h}''(x) + b y_{\rm h}'(x) + c y_{\rm h}(x) \;=\; 0. \tag{2.5.2}$$

Thus, $y_{\rm h}(x)$ has the form:

$$y_{\rm h}(x) \;=\; A y_1(x) + B y_2(x),$$

where:

- A, B are free parameters (independent of x, set by boundary conditions);

- $y_1(x)$, $y_2(x)$ are *independent* functions (one not a multiple of the other) that each solve Eq. (2.5.2) individually.

Theorem 2.4. *The* general *solution to Eq. (2.5.1) is:*

$$y(x) \;=\; y_{\rm h}(x) + y_{\rm p}(x) \;=\; A y_1(x) + B y_2(x) + y_{\rm p}(x). \tag{2.5.3}$$

Proof. Substitute Eq. (2.5.3) into LHS of Eq. (2.5.1) to get:

$$\begin{aligned}
& a\big(A y_1'' + B y_2'' + y_{\rm p}''\big) + b\big(A y_1' + B y_2' + y_{\rm p}'\big) + c\big(A y_1 + B y_2 + y_{\rm p}\big) \\
&\quad = A\big(\underbrace{a y_1'' + b y_1' + c y_1}_{0;\ \text{solves Eq. (2.5.2)}}\big) + B\big(\underbrace{a y_2'' + b y_2' + c y_2}_{0;\ \text{solves Eq. (2.5.2)}}\big) + \underbrace{a y_{\rm p}'' + b y_{\rm p}' + c y_{\rm p}}_{d(x);\ \text{solves Eq. (2.5.1)}} \\
&\qquad = d(x) \;=\; \text{RHS of Eq. (2.5.1)}.
\end{aligned}$$

Thus, Eq. (2.5.3) solves Eq. (2.5.1). It is the *general solution* because it has two free parameters, A, B, to be set by boundary conditions. $\square$

Thus, strategy for solving an inhomogeneous equation like Eq. (2.5.1):

1. Find *general* solution to homogeneous equation, Eq. (2.5.2), $y_\mathrm{h}(x)$.
2. Find *any* solution to Eq. (2.5.1), $y_\mathrm{p}(x)$, by:
   - inspection;
   - a clever substitution;
   - *variation of parameters* (PHYS 3200).
3. Construct general solution to Eq. (2.5.1), $y(x) = y_\mathrm{h}(x) + y_\mathrm{p}(x)$.
4. Apply boundary conditions to find *specific* solution.

Tutorial 4.3

Problem 4 For the underdamped harmonic oscillator described in the last tutorial, we found the damping coefficient to be $\gamma = \frac{1}{10}\ln 2$ and the natural oscillation frequency to be $\omega_\mathrm{d} \sim 10.0\,\mathrm{rad\,s^{-1}}$.

a) What is the phase offset of the oscillator, θ_0?

b) What is the quality factor, Q_d?

c) Approximately how many complete oscillations has the oscillator undergone by the time the amplitude falls to $x = x_0/2 = 2.00\,\mathrm{cm}$?

Problem 5 Solve by "inspection", "direct integration", and/or "trial exponentials" the linear, second-order, inhomogeneous ODE:

$$y''(x) + 4y(x) \;=\; f(x), \tag{1}$$

for boundary conditions $y(0) = -1$ and $y'(0) = 1$, where:

a) $f(x) = -2$;

b) $f(x) = 4e^{2x}$.

As discussed in §2.5 of the course notes, to solve such an *inhomogeneous* $[f(x) \neq 0]$ equation with boundary conditions, you must:

1. find two linearly independent solutions to the *homogeneous* equation [with $f(x) = 0$]; call these $y_1(x)$ and $y_2(x)$;

2. construct the general solution to the homogeneous equation,

$$y_\mathrm{h}(x) \;=\; Ay_1(x) + By_2(x),$$

   where A and B are constants;

3. by inspection or trial exponentials, find a *particular* solution, $y_\mathrm{p}(x)$ [*anything* that solves equation (1)];

4. write down the general solution to equation (1),

$$y(x) \;=\; y_\mathrm{h}(x) + y_\mathrm{p}(x);$$

5. and finally, apply boundary conditions to evaluate A and B.

Lesson 12

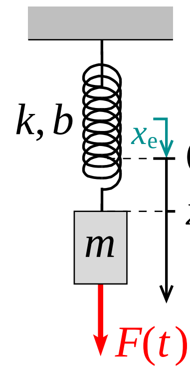
This lesson brings us to our most general case of harmonic motion: driven, damped, harmonic oscillation. Here we:

1. derive the inhomogeneous second order ODE that describes a driven, damped oscillator;

2. solve this ODE for certain initial conditions to find the *amplitude* and *phase* of the oscillator as functions of driving frequency; and

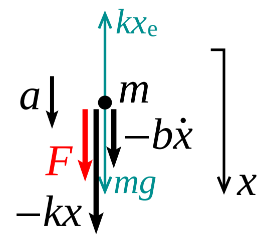
3. find the position of the oscillator as a function of time, noting the *transient* (homogeneous) and *steady-state* (particular) portions of the solution.

2.6 Forced harmonic motion (FC §3.6)



Apply force, $F(t) = F_0 \cos \omega t$, to m, where ω is the *driving frequency* (different from ω_0, ω_d).

From FBD and Newton's 2[nd] law,



$$\sum F = ma \quad \Rightarrow \quad F(t) + (-kx) + (-b\dot{x}) = m\ddot{x}$$

$$\Rightarrow \quad m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t$$

$$\Rightarrow \quad \boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t,} \tag{2.6.1}$$

an inhomogeneous 2[nd] order linear ODE.

We've already solved homogeneous version of Eq. (2.6.1) (*e.g.*, Eq. 2.4.2):

$$x_h(t) = e^{-\gamma t}\left(Ce^{i\omega_d t} + De^{-i\omega_d t}\right), \tag{2.6.2}$$

where: $\gamma = \dfrac{b}{2m}$ = damping coefficient;

$\omega_0 = \sqrt{\dfrac{k}{m}}$ = frequency of undamped oscillator;

$\omega_d = iq = \sqrt{\omega_0^2 - \gamma^2}$ = frequency of underdamped oscillator;

C, D = two free parameters set by initial conditions.

We now need a particular solution, $x_p(t)$, to Eq. (2.6.1)...

If $\omega = 0$, $F = F_0$ (constant) and "by inspection", $x_p = \dfrac{F_0}{k}$ solves Eq. (2.6.1).

$$\Rightarrow \quad x(t) = x_h(t) + x_p = e^{-\gamma t}\left(Ce^{i\omega_d t} + De^{-i\omega_d t}\right) + \frac{F_0}{k},$$

would be the general solution to Eq. (2.6.1).

Exercise: Show that for $\omega \neq 0$, $\dfrac{F_0 \cos \omega t}{k}$ is *not* a solution to Eq. (2.6.1).

So, for $\omega \neq 0$ we have to be more clever.

Since $F \propto \cos \omega t$, we might guess $x_{\mathrm{p}} \propto \cos \omega t$ as well. Since force and displacement are generally "out of phase", choose,

$$x_{\mathrm{p}}(t) \;=\; A \cos(\omega t - \phi). \tag{2.6.3}$$

If this solves Eq. (2.6.1), then,

$$-A\omega^2 \cos(\omega t - \phi) - 2\gamma A \omega \sin(\omega t - \phi) + \omega_0^2 A \cos(\omega t - \phi) \;=\; \frac{F_0}{m} \cos \omega t,$$

from which we should be able to solve for constants A and ϕ.

If our guess is wrong, algebra will lead to nonsense (*e.g.*, A, ϕ depend on t).

So, expanding out trig functions,

$$\begin{aligned}
&(\omega_0^2 - \omega^2) A (\cos \omega t \cos \phi + \sin \omega t \sin \phi) \\
&\qquad\qquad - 2\gamma\omega A (\sin \omega t \cos \phi - \cos \omega t \sin \phi) \;=\; \frac{F_0}{m} \cos \omega t \\
\Rightarrow \quad & \left[(\omega_0^2 - \omega^2) A \cos \phi + 2\gamma\omega A \sin \phi - \frac{F_0}{m} \right] \cos \omega t \\
&\qquad\qquad = \left[2\gamma\omega A \cos \phi - (\omega_0^2 - \omega^2) A \sin \phi \right] \sin \omega t.
\end{aligned}$$

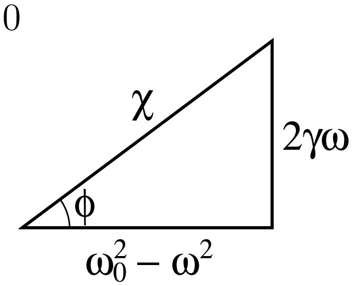
Quantities in [ ] are constant $\Rightarrow \cos \omega t \propto \sin \omega t$: impossible!

Only "way out" is for constants to be zero. Thus,

$$2\gamma\omega \cancel{A} \cos \phi - (\omega_0^2 - \omega^2) \cancel{A} \sin \phi \;=\; 0$$

$$\Rightarrow \quad \tan \phi \;=\; \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \tag{2.6.4}$$

$$\Rightarrow \quad \sin \phi \;=\; \frac{2\gamma\omega}{\chi}; \quad \cos \phi \;=\; \frac{\omega_0^2 - \omega^2}{\chi}, \tag{2.6.5}$$



where,

$$\chi \;=\; \sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}. \tag{2.6.6}$$

Next, $(\omega_0^2-\omega^2)A\cos\phi+2\gamma\omega A\sin\phi-\dfrac{F_0}{m}=0$

$$\Rightarrow \quad A \;=\; \frac{F_0/m}{(\omega_0^2-\omega^2)\cos\phi+2\gamma\omega\sin\phi} \;=\; \frac{\chi F_0/m}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

$$\Rightarrow \quad \boxed{A(\omega) \;=\; \frac{\chi F_0}{m\chi^2} \;=\; \frac{F_0}{m\chi} \;=\; \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}},} \tag{2.6.7}$$

using Eq. (2.6.5), (2.6.6).

A, ϕ independent of t (depend on ω) $\Rightarrow$ particular solution Eq. (2.6.3) OK!

General solution to Eq. (2.6.1) is homogeneous + particular solutions:

$$x(t) \;=\; e^{-\gamma t}\left(Ce^{i\omega_\mathrm{d}t}+De^{-i\omega_\mathrm{d}t}\right)+A\cos(\omega t-\phi), \tag{2.6.8}$$

with A (amplitude), ϕ (phase) given by Eq. (2.6.7), (2.6.4).

Only after $x_\mathrm{p}(t)$ is included are initial conditions applied to evaluate C, D.

From Eq. (2.6.8),

$$\begin{aligned}\dot{x}(t) \;=\; &-\gamma e^{-\gamma t}\left(Ce^{i\omega_\mathrm{d}t}+De^{-i\omega_\mathrm{d}t}\right)+i\omega_\mathrm{d}e^{-\gamma t}\left(Ce^{i\omega_\mathrm{d}t}-De^{-i\omega_\mathrm{d}t}\right)\\ &-A\omega\sin(\omega t-\phi).\end{aligned}$$

Setting $x(0)=0$ and $\dot{x}(0)=0$, we get:

$$C+D+A\cos\phi \;=\; 0; \qquad -\gamma(\underbrace{C+D}_{-A\cos\phi})+i\omega_\mathrm{d}(C-D)+A\omega\sin\phi \;=\; 0.$$

$$\Rightarrow \quad \begin{cases} C+D \;=\; -A\cos\phi \\ C-D \;=\; -A\,\dfrac{\gamma\cos\phi+\omega\sin\phi}{i\omega_\mathrm{d}} \end{cases}$$

Adding and subtracting these, we get:

$$\left.\begin{aligned} C &= -A\left(\frac{\cos\phi}{2} + \frac{\gamma\cos\phi + \omega\sin\phi}{2i\omega_{\rm d}}\right); \\ D &= -A\left(\frac{\cos\phi}{2} - \frac{\gamma\cos\phi + \omega\sin\phi}{2i\omega_{\rm d}}\right). \end{aligned}\right\} \tag{2.6.9}$$

Substituting Eq. (2.6.9) into (2.6.8), we get:

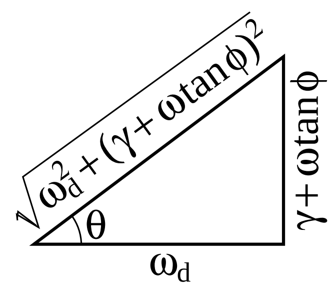
$$\begin{aligned} x(t) &= -Ae^{-\gamma t}\left(\cos\phi\,\frac{e^{i\omega_{\rm d}t} + e^{-i\omega_{\rm d}t}}{2} + \frac{\gamma\cos\phi + \omega\sin\phi}{\omega_{\rm d}}\,\frac{e^{i\omega_{\rm d}t} - e^{-i\omega_{\rm d}t}}{2i}\right) \\ &\quad + A\cos(\omega t - \phi) \\ &= \underbrace{-Ae^{-\gamma t}\left(\cos\phi\cos\omega_{\rm d}t + \frac{\gamma\cos\phi + \omega\sin\phi}{\omega_{\rm d}}\,\sin\omega_{\rm d}t\right)}_{x_{\rm h}(t);\ \textit{transient}\ \text{term}} + \underbrace{A\cos(\omega t - \phi)}_{x_{\rm p}(t);\ \textit{steady-state}}, \end{aligned}$$

using identities (2.4.7) (based on Euler's formula).

As $t \to \infty$, *transient term*[12] $\to 0$, leaving only *steady-state* (*asymptotic*) *term*.

To simplify $x_{\rm h}(t)$, first write:

$$\begin{aligned} x_{\rm h}(t) &= -Ae^{-\gamma t}\cos\phi\left(\cos\omega_{\rm d}t + \underbrace{\frac{\gamma + \omega\tan\phi}{\omega_{\rm d}}}_{\equiv\,\tan\theta}\sin\omega_{\rm d}t\right) \\ &= -Ae^{-\gamma t}\,\frac{\cos\phi}{\cos\theta}\left(\cos\omega_{\rm d}t\cos\theta + \sin\omega_{\rm d}t\sin\theta\right) \\ &= -Ae^{-\gamma t}\,\frac{\cos\phi}{\cos\theta}\,\cos(\omega_{\rm d}t - \theta) \end{aligned}$$



- θ = "phase lag" of transient term; ϕ = "phase lag" of steady state term.
- for $\omega = 0$ (no driving frequency), $\theta = \theta_0$ (Eq. 2.4.12).
- transient term oscillates with natural oscillator frequency, $\omega_{\rm d}$

[12]The text doesn't bother with the transient solution.

- steady-state term oscillates with driving force frequency, ω.

In the tutorial, we'll show that $\dfrac{\cos\phi}{\cos\theta} = \dfrac{\omega_0}{\omega_\mathrm{d}}$.

Thus, final form for our solution to Eq. (2.6.1) is:

$$\boxed{x(t) \;=\; -Ae^{-\gamma t}\,\frac{\omega_0}{\omega_\mathrm{d}}\,\cos(\omega_\mathrm{d}t-\theta) + A\cos(\omega t-\phi),} \tag{2.6.10}$$

where,

$$\tan\phi \;=\; \frac{2\gamma\omega}{\omega_0^2-\omega^2} \qquad \text{and} \qquad \tan\theta \;=\; \frac{\gamma+\omega\tan\phi}{\omega_\mathrm{d}} \;=\; \frac{\gamma}{\omega_\mathrm{d}}\,\frac{\omega_0^2+\omega^2}{\omega_0^2-\omega^2},$$

are phases of steady-state and transient terms.

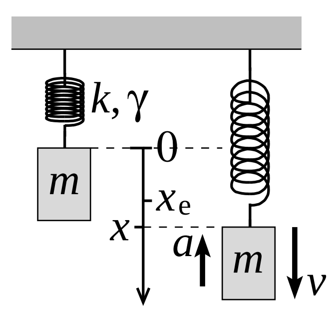
- transient term same as Eq. (2.4.13; unforced, underdamped) with A, θ depending on ω.
- $x_\mathrm{h}(t)$ is system's initial response to being "disturbed" by $F(t)$
- as $x_\mathrm{h}(t)$ dies away, system oscillates at frequency of $F(t)$, with phase lag ϕ (inertia of system)

How's your algebra? By direct substitution, show Eq. (2.6.10) solves (2.6.1). Show also that $x(0) = 0$ and $\dot{x}(0) = 0$ as required by initial conditions.

Show on-line handout `forced.pdf`.

Demo: damped, forced SHO

Example 2.10. A mass, m, on a vertical spring, k, γ, is released from rest when spring is in relaxed state. Find $x(t)$.



Solution: Here, $F = mg =$ constant and $\omega = 0$. Thus, from Eq. (2.6.10),

$$x(t) = -Ae^{-\gamma t}\frac{\omega_0}{\omega_d}\cos(\omega_d t - \theta) + A\cos\phi$$

$$\Rightarrow \quad \boxed{x(t) = \frac{mg}{k}\left(1 - e^{-\gamma t}\frac{\omega_0}{\omega_d}\cos(\omega_d t - \theta_0)\right),} \qquad (2.6.11)$$

since, for $\omega = 0$, $\phi = \sin^{-1}(2\gamma\omega/\chi) = 0$, and

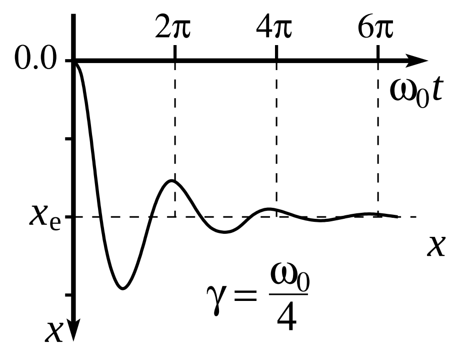
$$A = \frac{F_0}{k} = \frac{mg}{k} \quad \text{(Eq. 2.6.7, 2.6.6), and}$$

$$\tan\theta = \frac{\gamma + \omega\tan\phi}{\omega_d} = \frac{\gamma}{\omega_d} = \tan\theta_0 \quad \text{(Eq. 2.4.12).}$$

Note that at $t = 0$,

$$x(0) = \frac{mg}{k}\left(1 - \frac{\omega_0}{\omega_d}\underbrace{\cos(\theta_0)}_{\omega_d/\omega_0}\right) = 0, \quad \text{and}$$

$$\lim_{t\to\infty} x(t) = \frac{mg}{k} = x_e,$$



equilibrium point of spring under weight of m (Eq. 2.2.8), as expected.

For $0 < t < \infty$, oscillations about x_e gradually die down because of $e^{-\gamma t}$.

Tutorial 5.1

Problem 1 Show that,

$$\frac{\cos\phi}{\cos\theta} = \frac{\omega_0}{\omega_d},$$

where θ is the "phase lag" of transient term, ϕ is the "phase lag" of steady state term, and where ω_0 and ω_d are, respectively, the natural undamped and damped oscillation frequencies.

Problem 2 (FC 3.11) A mass m moves along the x-axis subject to a restoring force $F_r = -\frac{17}{2}\beta^2 mx$ and a drag force $F_d = -3\beta m\dot{x}$, where x is the distance from the origin and β is a constant. In addition, a driving force, $F = mA\cos\omega t$ is applied to the particle along the x-axis, where A is another constant.

a) What value of ω results in steady-state oscillations about the origin ($x = 0$) with maximum amplitude?

b) For the value of ω found in part a, what is the maximum amplitude?

Lesson 13

This lesson concludes our discussion on oscillators (for now!) by introducing *resonance*. Here, we:

1. extremise the amplitude as a function of driving frequency, $A(\omega)$, derived in the previous lesson, to find the *resonant frequency and amplitude*, ω_r and A_max;
2. examine cases where the driving frequency is much less, much more, or about the same as the resonant frequency; and
3. define and use the *sharpness of resonance*.

This lesson concludes with a discussion on *dimensional analysis* using the on-line handout.

2.6.1 Resonance

Amplitude and phase of forced oscillator given by:

$$A(\omega) = \frac{F_0/m}{\chi} = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}; \qquad (2.6.7,\ 2.6.6)$$

$$\phi(\omega) = \tan^{-1}\frac{2\gamma\omega}{\omega_0^2-\omega^2}. \qquad (2.6.4)$$

$A(\omega)$ is a maximum at *resonant frequency*, ω_r.

Maximum A occurs for minimum χ (equivalently χ^2), and we write:

$$\frac{d\chi^2}{d\omega} = \frac{d}{d\omega}\left[\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2\right] = 2(\omega_0^2-\omega^2)(-2\omega)+8\gamma^2\omega = 0$$

$$\Rightarrow \quad \omega_0^2-\omega_\mathrm{r}^2 = 2\gamma^2 \quad \Rightarrow \quad \boxed{\omega_\mathrm{r} = \sqrt{\omega_0^2-2\gamma^2},} \qquad (2.6.12)$$

- resonant frequency $\neq$ natural oscillation frequency, $\omega_\mathrm{d} = \sqrt{\omega_0^2-\gamma^2}$!
- $\omega_\mathrm{r}\in\mathbb{R} \Rightarrow \omega_0^2-2\gamma^2>0 \Rightarrow \gamma<\frac{\omega_0}{\sqrt{2}}$. If $\gamma>\frac{\omega_0}{\sqrt{2}}$, there is no resonance.

To find resonant amplitude,

$$A_\mathrm{max} = A(\omega_\mathrm{r}) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega_\mathrm{r}^2)^2+4\gamma^2\omega_\mathrm{r}^2}}$$

$$= \frac{F_0/m}{\sqrt{(\cancel{\omega_0^2}-\cancel{\omega_0^2}+2\gamma^2)^2+4\gamma^2(\omega_0^2-2\gamma^2)}} = \frac{F_0/m}{\sqrt{4\gamma^4+4\gamma^2\omega_0^2-8\gamma^4}}$$

$$\Rightarrow \quad \boxed{A_\mathrm{max}(\gamma) = \frac{F_0/m}{2\gamma\sqrt{\omega_0^2-\gamma^2}} = \frac{F_0}{2m\gamma\,\omega_\mathrm{d}}.} \qquad (2.6.13)$$

As a function of ω_r, since $\gamma^2=\frac{1}{2}(\omega_0^2-\omega_\mathrm{r}^2)$, $\omega_0^2-\gamma^2=\frac{1}{2}(\omega_0^2+\omega_\mathrm{r}^2)$, we have,

$$A_\mathrm{max}(\omega_\mathrm{r}) = \frac{F_0/m}{\cancel{2}\sqrt{\frac{1}{\cancel{2}}(\omega_0^2-\omega_\mathrm{r}^2)}\sqrt{\frac{1}{\cancel{2}}(\omega_0^2+\omega_\mathrm{r}^2)}} = \frac{F_0}{m\sqrt{\omega_0^4-\omega_\mathrm{r}^4}}. \qquad (2.6.14)$$

Then, from Eq. (2.6.4), phase at resonance is:

$$\tan\phi_r = \frac{2\gamma\omega_r}{\omega_0^2 - \omega_r^2} = \frac{2\gamma\omega_r}{2\gamma^2} \quad \Rightarrow \quad \boxed{\tan\phi_r = \frac{\omega_r}{\gamma}.} \tag{2.6.15}$$

For weakly damped systems, $\gamma \to 0$ and $\phi_r \to \dfrac{\pi}{2}$.

Show on-line handout `resonance.pdf`.

Case 1: $\omega \ll \omega_r$ (*e.g.*, $\omega \to 0$)

- $A \to \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{k} \Rightarrow F_0$, k determine amplitude; γ, m, ω irrelevant.
- $\phi \to 0 \Rightarrow$ force and displacement in phase.

Case 2: $\omega \gg \omega_r$ (*e.g.*, $\omega \to \infty$)

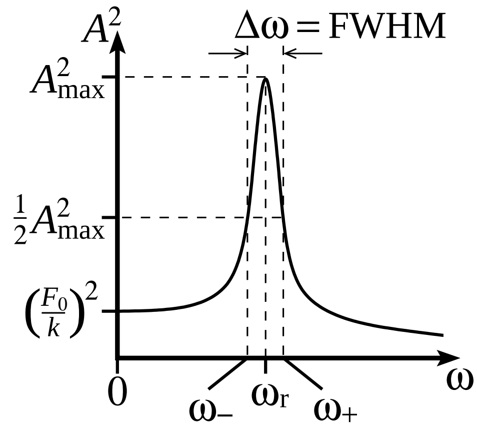
- $A \to \dfrac{F_0}{m\omega^2} \propto \omega^{-2} \Rightarrow F_0$, m, ω determine amplitude; γ, k irrelevant.
- m shaken back and forth at high ω as though spring weren't there
- $\phi \to \pi \Rightarrow$ displacement opposite in phase to force.

Demo: metre stick in phase (low ω) and out of phase (high ω)

Case 3: $\omega \sim \omega_r$ (near resonance)

- $A \to \dfrac{F_0}{b\,\omega_d} \propto \omega^{-1} \Rightarrow F_0$, $b = 2m\gamma$ determine amplitude, k irrelevant.
- $\phi \to \dfrac{\pi}{2} \Rightarrow x(t) \to A\cos\left(\omega t + \frac{\pi}{2}\right) = A\sin\omega t \Rightarrow v(t) = A\omega\cos\omega t$, and velocity is in phase with force.
- to push a child on a swing, use small pushes at bottom of swing where $x = 0$, v is a maximum, and power $= Fv$ is a maximum.

Sharpness of resonance and $Q_{\rm d}$



Sharpness of resonance defined as,

$$S = \frac{\omega_+ - \omega_-}{\omega_{\rm r}} = \frac{\Delta\omega}{\omega_{\rm r}}, \qquad (2.6.16)$$

where $\omega_\pm$ = frequencies where total *energy*, $\frac{1}{2}kA^2$, is half that at resonance; $\Delta\omega$ is *full-width-half-maximum* (FWHM) of profile.

To find $\omega_\pm$, from Eq. (2.6.7), (2.6.13) set,

$$A^2(\omega_\pm) = \frac{(F_0/m)^2}{(\omega_0^2 - \omega_\pm^2)^2 + 4\gamma^2\omega_\pm^2} = \frac{1}{2}A^2_{\rm max} = \frac{(F_0/m)^2}{8\gamma^2(\omega_0^2-\gamma^2)}$$

$$\Rightarrow \quad \omega_0^4 - 2\omega_0^2\omega_\pm^2 + \omega_\pm^4 + 4\gamma^2\omega_\pm^2 = 8\gamma^2\omega_0^2 - 8\gamma^4.$$

$$\Rightarrow \quad \omega_\pm^4 - 2\omega_\pm^2\underbrace{(\omega_0^2 - 2\gamma^2)}_{\omega_{\rm r}^2} + \underbrace{\omega_0^4 - 4\gamma^2\omega_0^2 + 4\gamma^4}_{(\omega_0^2-2\gamma^2)^2 = \omega_{\rm r}^4} = 4\gamma^2\underbrace{(\omega_0^2 - \gamma^2)}_{\omega_{\rm d}^2}.$$

$$\Rightarrow \quad \omega_\pm^4 - 2\omega_\pm^2\omega_{\rm r}^2 + \omega_{\rm r}^4 = (\omega_\pm^2 - \omega_{\rm r}^2)^2 = 4\gamma^2\omega_{\rm d}^2.$$

$$\Rightarrow \quad \omega_\pm^2 - \omega_{\rm r}^2 = \pm 2\gamma\omega_{\rm d} \quad \Rightarrow \quad \boxed{\omega_\pm^2 = \omega_{\rm r}^2 \pm 2\gamma\omega_{\rm d}.} \qquad (2.6.17)$$

Then we can write,

$$\omega_+^2 - \omega_-^2 = \underbrace{\frac{\omega_+ + \omega_-}{2}}_{\bar\omega \approx \omega_{\rm r}} 2\underbrace{(\omega_+ - \omega_-)}_{\Delta\omega} \approx 2\omega_{\rm r}\Delta\omega^{13}$$

$$= \cancel{\omega_{\rm r}^2} + 2\gamma\omega_{\rm d} - (\cancel{\omega_{\rm r}^2} - 2\gamma\omega_{\rm d}) = 4\gamma\omega_{\rm d},$$

where approximation best for $\Delta\omega \ll \omega_{\rm r}$ (narrow peaks).

Thus, FWHM given by,

$$\Delta\omega \approx 2\gamma\frac{\omega_{\rm d}}{\omega_{\rm r}} = \frac{\omega_{\rm d}^2}{Q_{\rm d}\omega_{\rm r}} \approx \frac{\omega_0}{Q_{\rm d}} \approx 2\gamma,$$

[13] The expression $\Delta\omega^2 = 2\bar\omega\Delta\omega$ should remind students of the rule for differentials: $d\omega^2 = 2\omega d\omega$.

for $\gamma \ll \omega_0$, and $Q_\mathrm{d} = \dfrac{\omega_\mathrm{d}}{2\gamma}$ (quality factor; Eq. 2.4.16). Then Eq. (2.6.16) $\Rightarrow$

$$\boxed{S = \frac{\Delta\omega}{\omega_\mathrm{r}} \approx \frac{\Delta\omega}{\omega_0} \approx \frac{1}{Q_\mathrm{d}},} \tag{2.6.18}$$

and Q_d is an inverse measure of resonance sharpness.

See examples 3.6.1 and 3.6.2 in text.

Show handout `dim_analysis.pdf`, *which uses equations from this part.*

Tutorial 5.2

Problem 2 (FC 3.14) An oscillator consists of a spring with spring constant, k, and a damping coefficient, γ, equal to one half the critical value. If the undamped frequency of the oscillator is ω_0 and it is driven by a force $F_0 \cos(2\omega_0 t)$, find:

a) the resonant frequency, ω_r;

b) the phase, ϕ, between the driving force and the oscillator displacement; and

c) the steady state amplitude.

Lesson 14

This lesson begins our discussion on the motion of a single particle in 3-D. We start by reviewing the elements of *vector calculus* including:

1. *partial derivatives* and the *nabla* (∇);
2. the *gradient* of a scalar function, and the *divergence* and *curl* of a vector function; and
3. second derivatives involving the nabla,

and complete the lesson by rederiving the work-kinetic theorem and the principle of conservation of mechanical energy in 3-D. In so doing, we shall define what it means for a force to be *conservative*.

Part III: Motion of a particle in 3-D

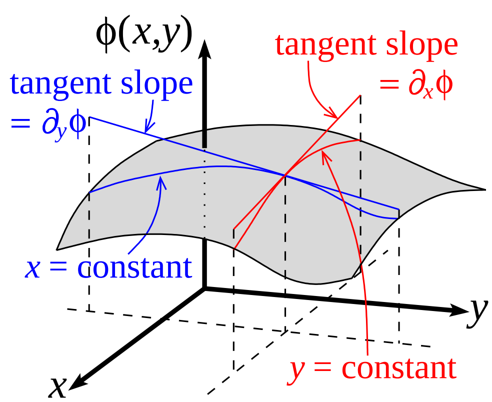
Reading assignment: Chapter 4

3.1 Elements of vector calculus

The *partial derivatives* of a *bivariate, scalar* function $\phi(x, y)$ are defined as:

$$\frac{\partial \phi}{\partial x} \equiv \partial_x \phi \equiv \lim_{h \to 0} \frac{\phi(x+h, y) - \phi(x, y)}{h};$$

$$\frac{\partial \phi}{\partial y} \equiv \partial_y \phi \equiv \lim_{h \to 0} \frac{\phi(x, y+h) - \phi(x, y)}{h}.$$



$\partial_x \phi$ = rate of change of ϕ with respect to x holding y constant;

etc. for $\partial_y \phi$ and $\partial_z \phi$ if $\phi(x, y, z)$ is *trivariate.*

The *nabla* or *del operator* is the *ordered triple*, $\nabla \equiv (\partial_x, \partial_y, \partial_z)$.

- an operator is neither a function nor a number; it cannot be "evaluated"
- an operator is a set of instructions to be applied to a function:
  - integrate
  - differentiate
  - multiply by a then add b, *etc.*

The nabla operates on a function by differentiating it in one of several ways:

1. *Gradient* of a scalar function, $\phi(x, y, z)$, is a vector:

$$\nabla \phi = (\partial_x \phi, \partial_y \phi, \partial_z \phi). \tag{3.1.1}$$

Consider a differential of the function $\phi(x, y, z)$:

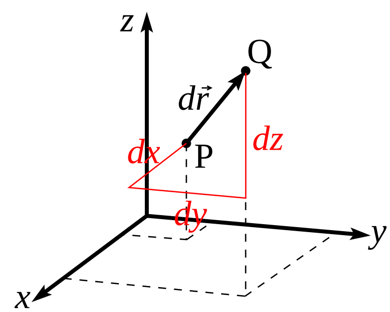
$$d\phi \;=\; \partial_x\phi\, dx + \partial_y\phi\, dy + \partial_z\phi\, dz \;=\; \nabla\phi \cdot d\vec{r}, \tag{3.1.2}$$

using the chain rule. Thus, going from P to Q along vector $d\vec{r} = (dx, dy, dz)$, net change in ϕ, $d\phi$, is:

change in ϕ as one moves along dx: $\partial_x\phi\, dx$,

$+$ change in ϕ as one moves along dy: $\partial_y\phi\, dy$,

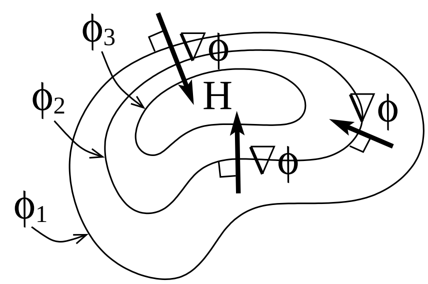
$+$ change in ϕ as one moves along dz: $\partial_z\phi\, dz$.



Total change, $d\phi$, is:

- maximum when $d\vec{r} \parallel \nabla\phi \;\Rightarrow\; \nabla\phi$ points in direction of *steepest ascent/descent*;

- 0 when $d\vec{r} \perp \nabla\phi \;\Rightarrow\; \nabla\phi \perp$ to lines (surfaces) of constant ϕ: "contours".



(*Ski-hill analogy*)

2. *Divergence* of a vector function, $\vec{A}(x, y, z)$, is a scalar:

$$\nabla \cdot \vec{A} \;=\; \partial_x A_x + \partial_y A_y + \partial_z A_z. \tag{3.1.3}$$

If $\nabla \cdot \vec{A} = 0$, $\vec{A}$ is said to be *solenoidal*.

3. *Curl* of a vector function, $\vec{A}(x, y, z)$, is a vector:

$$\nabla \times \vec{A} \;=\; (\partial_y A_z - \partial_z A_y)\,\hat{\imath} + (\partial_z A_x - \partial_x A_z)\,\hat{\jmath} + (\partial_x A_y - \partial_y A_x)\,\hat{k} \tag{3.1.4}$$

If $\nabla \times \vec{A} = 0$, $\vec{A}$ is said to be *irrotational*.

4. *Second derivatives*

Let f, $\vec{A}$ be a scalar, vector function of the coordinates. Then,

$$\begin{aligned} \nabla \times (\nabla f) &= 0; \qquad (3.1.5)\\ \nabla \cdot (\nabla \times \vec{A}) &= 0;\\ \nabla \cdot (\nabla f) &= \nabla^2 f;\\ \nabla(\nabla \cdot \vec{A}) &= \nabla^2 \vec{A}. \end{aligned}$$

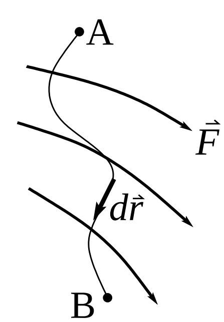
3.2 *W-K* theorem, conservation of energy (FC §4.1, 4.2, 4.6)

Newton's second law with constant m:

$$\vec{F}(\vec{r}, \vec{v}, t) = m\frac{d\vec{v}}{dt}$$

$$\Rightarrow \quad \vec{F} \cdot \vec{v} = m\frac{d\vec{v}}{dt} \cdot \vec{v} = m\frac{1}{2}\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = \frac{dK}{dt}$$

Thus, $\vec{F} \cdot \vec{v}\, dt = \vec{F} \cdot d\vec{r} = dK$. Integrating this along an arbitrary path joining points A and B, we get:



$$\boxed{\int_{\mathrm{A}}^{\mathrm{B}} \vec{F} \cdot d\vec{r} \equiv W_{\mathrm{AB}} = \int_{\mathrm{A}}^{\mathrm{B}} dK = K_{\mathrm{B}} - K_{\mathrm{A}},} \qquad (3.2.1)$$

where W_{AB} is *work* done by $\vec{F}$ along path from A to B.

This is the same *work-kinetic theorem* found for rectilinear motion Eq. (1.7.7):

Work done by all forces on m is the net change in its kinetic energy.

In 1-D (Eq. 1.7.8), if $F = F(x)$, $\exists\, U(x)$ (potential energy), such that:

$$W_{\mathrm{AB}} = \int_{x_{\mathrm{A}}}^{x_{\mathrm{B}}} F(x)\, dx = U_{\mathrm{A}} - U_{\mathrm{B}}; \qquad F(x) = -\frac{dU(x)}{dx}. \qquad (3.2.2)$$

In 3-D, W_{AB} in Eq. (3.2.1) is a *path integral*, not a 1-D integral like Eq. (3.2.2).

In particular, $\vec{F} = \vec{F}(\vec{r})$ is necessary[14] but insufficient to ensure a potential energy function, $U(\vec{r})$, exists such that:

$$\vec{F}(\vec{r}) \;=\; -\nabla U(\vec{r}), \tag{3.2.3}$$

the 3-D analogue of Eq. (3.2.2).

Theorem 3.1. *Suppose* $\vec{F} = \vec{F}(\vec{r})$*. Then,* $\nabla \times \vec{F} = 0 \iff$ *there exists a scalar function,* $U(\vec{r})$*, such that* $\vec{F} = -\nabla U$*.*

An irrotational force $(\nabla \times \vec{F} = 0)$ *is said to be* conservative*.*

Proof. Such theorems with an "if and only if" symbol ($\iff$) require both "directions" to be proven.

$\Leftarrow$ *proof*: Here, we assume $\vec{F} = -\nabla U$, and prove $\nabla \times \vec{F} = 0$. Thus,

$$\nabla \times \vec{F} \;=\; -\nabla \times (\nabla U) \;=\; 0,$$

using identity Eq. (3.1.5).

$\Rightarrow$ *proof*: Here, we assume $\nabla \times \vec{F} = 0$, and prove $\vec{F} = -\nabla U$ for some $U(\vec{r})$.

This proof requires *Stokes' Theorem*; see handout `conservative.pdf`. □

$\vec{F}$ conservative $(\nabla \times \vec{F} = 0) \Rightarrow \vec{F} = -\nabla U$. Using Eq. (3.1.2; $\nabla\phi \cdot \mathrm{d}\vec{r} = d\phi$),

$$W_F \;=\; \int_{\mathrm{A}}^{\mathrm{B}} \vec{F} \cdot d\vec{r} \;=\; -\int_{\mathrm{A}}^{\mathrm{B}} \nabla U \cdot d\vec{r} \;=\; -\int_{\mathrm{A}}^{\mathrm{B}} dU \;=\; U_{\mathrm{A}} - U_{\mathrm{B}}$$

$\Rightarrow$ W_F depends only on end points, A, B, not on path taken between them.

[14] If $U(\vec{r})$ is to be an energy of *position*, it cannot depend on $\vec{v}$ or t, and thus neither can $\vec{F}$.

But, from Eq. (3.2.1),

$$\int_A^B \vec{F} \cdot d\vec{r} = K_B - K_A.$$

$$\Rightarrow \quad U_A - U_B = K_B - K_A \quad \Rightarrow \quad U_A + K_A = U_B + K_B.$$

Define *mechanical energy*, $E = U + K$. Then $E_A = E_B$ and we have the *law of conservation of energy*:

If $\vec{F}$ is conservative, mechanical energy is conserved.

Example 3.1. Is $\vec{F} = b(y, x, 0)$ conservative? If so, find $U(\vec{r})$.

Solution:
$$\nabla \times \vec{F} = \big(\partial_y \cancelto{0}{F_z} - \cancelto{0}{\partial_z F_y}, \cancelto{0}{\partial_z F_x} - \partial_x \cancelto{0}{F_z}, \partial_x F_y - \partial_y F_x\big)$$
$$= \big(0, 0, \partial_x(bx) - \partial_y(by)\big) = (0, 0, 0),$$

and $\vec{F}$ is conservative.

To find $U(\vec{r})$, "integrate" Eq. (3.2.3). Thus,

$$F_x = by = -\partial_x U \quad \Rightarrow \quad U = -\int by\, dx = -byx + f(y);$$

$$F_y = bx = -\partial_y U \quad \Rightarrow \quad U = -\int bx\, dy = -bxy + g(x),$$

where "constants" of integration, f and g, are, in fact, functions of independent variable(s) not integrated. Both solutions must be valid, and thus,

$$-\cancel{byx} + f(y) = -\cancel{bxy} + g(x) \quad \Rightarrow \quad f(y) = g(x).$$

Functions of independent arguments are equal only when each is the same constant: $f(y) = g(x) = U_0$, say. Thus,

$$U(\vec{r}) = -bxy + U_0,$$

where U_0 could be set to zero, if convenient. □

Example 3.2. Is $\vec{G} = b(-y, x, 0)$ conservative? If so, find $U(\vec{r})$.

Solution: This time,

$$\nabla \times \vec{G} \;=\; \big(0, 0, \partial_x(bx) - \partial_y(-by)\big) \;=\; (0, 0, 2b),$$

and $\vec{G}$ is not conservative.

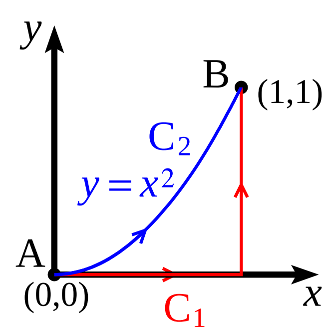
If we try to find $U(\vec{r})$ despite $\nabla \times \vec{G} \neq 0$, then on "integrating" Eq. (3.2.3),

$$G_x \;=\; -by \;=\; -\partial_x U \;\;\Rightarrow\;\; U \;=\; byx + f(y);$$
$$G_y \;=\; bx \;=\; -\partial_y U \;\;\Rightarrow\;\; U \;=\; -bxy + g(x)$$
$$\Rightarrow\;\; byx + f(y) \;=\; -bxy + g(x) \;\;\Rightarrow\;\; g(x) \;=\; 2bxy + f(y).$$

Reductio ad absurdum (g doesn't depend upon y) $\Rightarrow$ $U(\vec{r})$ does not exist. □

Example 3.3. Evaluate work done by each of $\vec{F}$ and $\vec{G}$ along paths, C_1, C_2.

Solution: Along path C_1:



$$\int_{C_1} \vec{F}\cdot d\vec{r} \;=\; b\int_{C_1} (y\,dx + x\,dy)$$
$$=\; b\int_0^1 y\,dx\Big|_{y=0} + b\int_0^1 x\,dy\Big|_{x=1}$$
$$=\; 0 + by\Big|_0^1 \;=\; b;$$

$$\int_{C_1} \vec{G}\cdot d\vec{r} \;=\; b\int_{C_1} (-y\,dx + x\,dy) \;=\; -b\int_0^1 y\,dx\Big|_{y=0} + b\int_0^1 x\,dy\Big|_{x=1} \;=\; b.$$

Then, along C_2 where $y = x^2 \;\Rightarrow\; dy = 2x\,dx$:

$$\int_{C_2} \vec{F}\cdot d\vec{r} \;=\; b\int_{C_2} (y\,dx + x\,dy) \;=\; b\int_0^1 (x^2dx + 2x^2dx) \;=\; 3b\int_0^1 x^2dx \;=\; b;$$

$$\int_{C_2} \vec{G}\cdot d\vec{r} \;=\; b\int_{C_2} (-y\,dx + x\,dy) \;=\; b\int_0^1 (-x^2dx + 2x^2dx) \;=\; b\int_0^1 x^2dx \;=\; \frac{b}{3}.$$

Evidently. W_F is path-independent, whereas W_G is not.

Tutorial 6.1

Problem 1 (FC 4.1) Find the forces corresponding to each of the following potential energy functions.

a) $U(x, y, z) = cxyz + C$

b) $U(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2 + C$

c) $U(x, y, z) = ce^{-(\alpha x + \beta y + \gamma z)}$

d) $U(r) = cr^n$ in spherical coordinates

Problem 2 Which of the forces in parts a and b is/are conservative, and in part c, find the value of α that makes the force conservative.

a) $\vec{F}(x, y, z) = \big(xy + z^2, y(2z + x), 2xz + y^2\big)$

b) $\vec{F}(x, y, z) = (\sin x\, e^z)\hat{\imath} + (z \sin y)\hat{\jmath} - (\cos x\, e^z + \cos y)\hat{k}$

c) $\vec{F}(x, y, z) = \big(\cosh z - x^2 e^{\alpha y}, -x^3 e^{\alpha y} + 2yz, x \sinh z + y^2\big)$

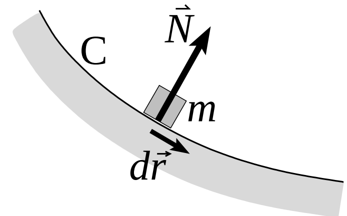
Lesson 15

This lesson continues our discussion on motion of a particle in 3-D by:

1. introducing *constrained motion* and the role of the normal force;
2. applying the W-K theorem and/or conservation of mechanical energy to problems with constrained motion;
3. defining *separable forces*; and
4. deriving the equations of motion for a projectile without air resistance.

<u>Constrained motion</u>

Mass m constrained to move along fixed path, C.



Since $\vec{N} \perp$ surface and $d\vec{r} \parallel$ surface,

$$\vec{N} \cdot d\vec{r} = 0 \quad \Rightarrow \quad W_N = \int \vec{N} \cdot d\vec{r} = 0,$$

and *normal force does no work.*

Further, $\vec{N} \perp d\vec{r} \Rightarrow \vec{N} \perp \dfrac{d\vec{r}}{dt} = \vec{v}$. If $\vec{F}$ = all forces on m other than $\vec{N}$,

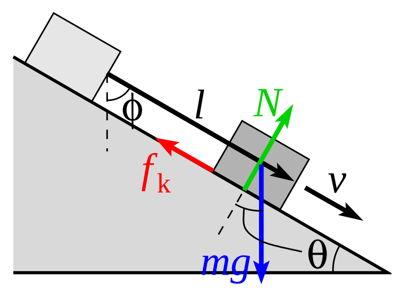
$$\vec{F} + \vec{N} = m\frac{d\vec{v}}{dt} \quad \Rightarrow \quad \vec{v} \cdot (\vec{F} + \vec{N}) = \vec{v} \cdot \vec{F} + \cancelto{0}{\vec{v} \cdot \vec{N}} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{dK}{dt}$$

$$\Rightarrow \quad \int dK = \int \vec{F} \cdot \vec{v}\, dt = \int \vec{F} \cdot d\vec{r} = -\int dU \quad \text{if } \nabla \times \vec{F} = 0.$$

$$\Rightarrow \quad K - K_{\mathrm{i}} = -U + U_{\mathrm{i}} \quad \Rightarrow \quad K + U = K_{\mathrm{i}} + U_{\mathrm{i}} = E,$$

and $\vec{N}$ *plays no role in conservation of mechanical energy.*

Example 3.4. Starting from rest, m slides down an inclined plane (θ) with kinetic coefficient of friction, μ_{k}. Use W-K theorem to find v after m slides distance l.



Solution: Total work done by forces on m is:

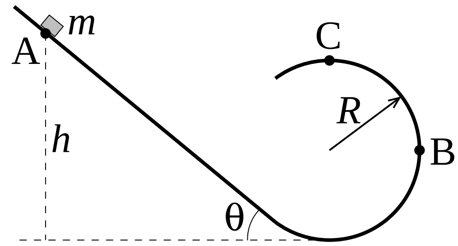
$$W = W_{mg} + W_{f_{\mathrm{k}}} + \cancelto{0}{W_N} = \int_0^l (m\vec{g} + \vec{f}_{\mathrm{k}}) \cdot d\vec{r} = \int_0^l mg\cos\phi\, dr - \int_0^l \mu_{\mathrm{k}} N\, dr$$

$$= mg\sin\theta\, l - \mu_{\mathrm{k}} mg\cos\theta\, l = \Delta K = K_{\mathrm{f}} - \cancelto{0}{K_{\mathrm{i}}} = \frac{1}{2}mv^2$$

$$\Rightarrow \quad v = \sqrt{2gl(\sin\theta - \mu_{\mathrm{k}}\cos\theta)},$$

identical result to Eq. (1.5.11). □

Example 3.5. m is released from rest at A from height h and slides along a frictionless track whose circular portion has radius R.



a) Find normal force track exerts on m at B.

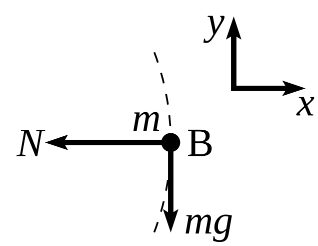
Solution: With no friction, mechanical energy is conserved:

$$E_A = K_A + U_A = 0 + mgh; \qquad E_B = K_B + U_B = \tfrac{1}{2}mv_B^2 + mgR$$

$$E_A = E_B \quad \Rightarrow \quad mgh = \tfrac{1}{2}mv_B^2 + mgR \quad \Rightarrow \quad v_B^2 = 2g(h - R). \quad (3.2.4)$$

Next, from FBD at B, in x-direction: $\sum F_x = ma_x \;\Rightarrow$

$$-N = -ma_{cp} = -m\frac{v_B^2}{R} = -m\frac{2g(h-R)}{R},$$



using Eq. (3.2.4). Thus,

$$\boxed{N = 2mg\left(\frac{h}{R} - 1\right).}$$

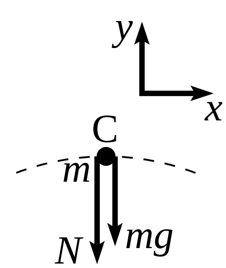
b) Find minimum h required for m to remain in contact with track at C.

Solution: Conserving mechanical energy,

$$E_A = E_C \quad \Rightarrow \quad mgh = \tfrac{1}{2}mv_C^2 + 2mgR \quad \Rightarrow \quad v_C^2 = 2g(h - 2R).$$

Next, consider FBD at C.

- m still follows circular track $\Rightarrow a_y = -\dfrac{v_C^2}{R}$ (otherwise, m is in a parabolic trajectory with $a_y = -g$).
- for *minimum* h, $N = 0$ (otherwise m faster and h higher than need be).



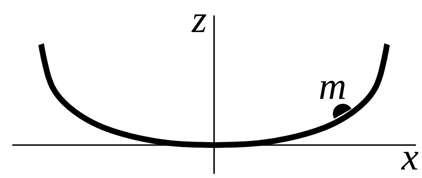
Thus, in y direction: $\sum F_y = ma_y \;\Rightarrow$

$$-\cancelto{0}{N} - mg \;=\; -m\frac{v_{\mathrm{C}}^2}{R} \;=\; -2mg\frac{h-2R}{R}$$

$$\Rightarrow \quad R \;=\; 2h - 4R \quad \Rightarrow \quad \boxed{h \;=\; \frac{5R}{2}.} \qquad \square$$

Example 3.6. (Ex. 4.6.2, pp. 178–9 in text.) A particle slides along a smooth *cycloid*, parameterised by $-\pi/4 \le \phi \le \pi/4$:

$$x \;=\; C(2\phi + \sin 2\phi); \quad z \;=\; C(1 - \cos 2\phi).$$



(C characterises shape of cycloid, not amplitude of oscillation.) Find period of oscillation.

Strategy: When asked to find a *period*, try finding an SHO equation:

$$\text{Forces} \;\Rightarrow\; \ddot{s} = -\omega_0^2 s; \quad \text{energy} \;\Rightarrow\; E = \tfrac{1}{2}m\dot{s}^2 + \tfrac{1}{2}ks^2, \text{ with } \omega_0^2 = \frac{k}{m}.$$

Start by conserving mechanical energy:

$$E \;=\; \frac{m}{2}(\dot{x}^2 + \dot{z}^2) + mgz \;=\; \text{constant}.$$

Now, $\dot{x} = 2C\dot{\phi}(1 + \cos 2\phi)$ and $\dot{z} = 2C\dot{\phi}\sin 2\phi$. Thus,

$$\begin{aligned} E \;&=\; \frac{m}{2}4C^2\dot{\phi}^2(1 + 2\cos 2\phi + \underbrace{\cos^2 2\phi + \sin^2 2\phi}_{1}) + mgC(\underbrace{1 - \cos 2\phi}_{2\sin^2\phi}) \\ &=\; 4mC^2\dot{\phi}^2(\underbrace{1 + \cos 2\phi}_{2\cos^2\phi}) + 2mgC\sin^2\phi \\ &=\; \frac{1}{2}m(4C\dot{\phi}\cos\phi)^2 + \frac{1}{2}\frac{mg}{4C}(4C\sin\phi)^2. \end{aligned}$$

Define $k = \dfrac{mg}{4C}$ and $s \equiv 4C\sin\phi \;\Rightarrow\; \dot{s} = 4C\dot{\phi}\cos\phi$. Then,

$$E \;=\; \tfrac{1}{2}m\dot{s}^2 + \tfrac{1}{2}ks^2 \;=\; \text{constant},$$

same form as SHO energy equation with position s! Thus (problem 4.23),

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{4C}} \quad \Rightarrow \quad T = \frac{2\pi}{\omega_0} = 4\pi\sqrt{\frac{C}{g}}.$$

Period is independent of amplitude of oscillation, s_0, and is *isochronus* (similar to a mass on a spring, unlike a large-amplitude pendulum). □

Separable forces

Separable forces have the form:

$$\vec{F}(\vec{r}, \dot{\vec{r}}, t) = F_x(x, \dot{x}, t)\,\hat{\imath} + F_y(y, \dot{y}, t)\,\hat{\jmath} + F_z(z, \dot{z}, t)\,\hat{k},$$

Not all forces are separable, and separable forces need not be conservative.

Separable forces can be solved by solving three independent 1-D problems:

$$F_x(x, \dot{x}, t) = m\ddot{x}; \qquad F_y(y, \dot{y}, t) = m\ddot{y}; \qquad F_z(z, \dot{z}, t) = m\ddot{z}.$$

Then, as concluded for 1-D, $\vec{F}$ is conservative if:

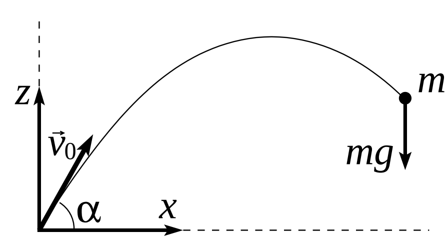
$$F_x = F_x(x); \qquad F_y = F_y(y); \qquad F_z = F_z(z). \qquad (3.2.5)$$

Note that components of a curl don't involve derivatives with respect to their own coordinate [*e.g.*, no x-derivatives in $(\nabla \times \vec{F})_x$].

Thus, if vector components are given by Eq. (3.2.5), $\nabla \times \vec{F} = 0$.

3.3 Projectile motion (FC §4.3)

3.3.1 No air resistance



A projectile, m, launched with velocity $\vec{v}_0$ at elevation angle α, moves in x-z (horizontal-vertical) plane under influence of gravity alone. Newton's 2$^{\text{nd}}$ law $\Rightarrow$

$$\sum \vec{F} \;=\; m\vec{g} \;=\; m\vec{a} \quad\Rightarrow\quad \vec{a} \;=\; \ddot{x}\hat{\imath} + \ddot{z}\hat{k} \;=\; g(-\hat{k}).$$

$$\Rightarrow \quad a_x \;=\; \ddot{x} \;=\; 0 \quad \text{and} \quad a_z \;=\; \ddot{z} \;=\; -g \qquad \text{(separable)}.$$

These are kinematics problems ($\vec{a}$ constant) and we can immediately write:

$$\boxed{x(t) \;=\; x_0 + \dot{x}_0 t; \quad \text{and} \quad z(t) \;=\; z_0 + \dot{z}_0 t - \frac{1}{2} g t^2,} \tag{3.3.1}$$

from Eq. (1.5.3). Combining these into a single vector equation gives:

$$\vec{r}(t) \;=\; (x_0 + \dot{x}_0 t)\hat{\imath} + (z_0 + \dot{z}_0 t - \tfrac{1}{2} g t^2)\hat{k} \;=\; \vec{r}_0 + \vec{v}_0 t - \tfrac{1}{2} g t^2 \hat{k}, \tag{3.3.2}$$

where $\vec{r}_0 = (x_0, z_0)$ and $\vec{v}_0 = (\dot{x}_0, \dot{z}_0) = v_0(\cos\alpha, \sin\alpha)$.

Finally, solving first of Eq. (3.3.1) for t, we get,

$$t \;=\; \frac{x - x_0}{\dot{x}_0} \;=\; \frac{x - x_0}{v_0 \cos\theta},$$

and substituting this into second of Eq. (3.3.1), we get *trajectory equation*:

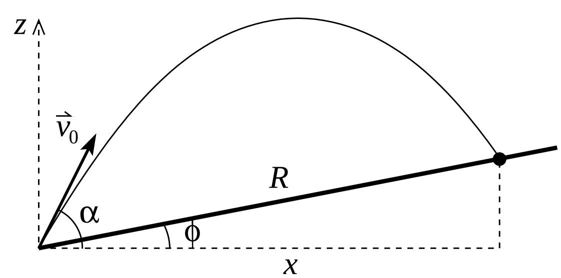
$$\boxed{z(x) - z_0 \;=\; (x - x_0)\tan\alpha - \frac{g(x - x_0)^2}{2v_0^2 \cos^2\!\alpha}.} \tag{3.3.3}$$

Tutorial 6.2

Problem 3 (FC 4.8) A gun is located at the bottom of a hill with constant inclination angle ϕ and aimed at a target up the hill.

a) If air resistance is negligible and α is the angle of elevation of the gun measured between the x-axis and $\vec{v}_0$, show that the range of the gun, R, up the hill is:

$$R \;=\; \frac{2v_0^2 \cos\alpha \sin(\alpha - \phi)}{g \cos^2\phi}.$$



b) What is the optimal angle of elevation, α, that maximises the range of the gun, and what is that maximum range?

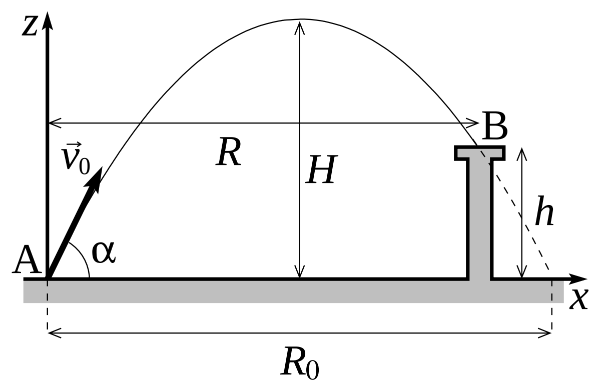
Lesson 16

This lesson concludes our discussion on projectile motion by:

1. presenting a detailed example of a projectile beginning and ending at two different levels;
2. examining projectiles with a simple model for *air resistance*; and
3. comparing the equations of motion for a projectile with and without air resistance.

Example 3.7. (F&C 4.10) Launcher A, launch speed v_0, is at $\vec{r}_0 = (0,0)$. Target B is on a tower of height h.

a) For a given α, at what distance, R, should tower be from A for projectile to reach B?



Solution: Set $(x_0, z_0) = (0,0)$, $x = R$, $z(R) = h$ in Eq. (3.3.3); solve for R:

$$h \;=\; R\tan\alpha - \frac{gR^2}{2v_0^2\cos^2\!\alpha}$$

$$\times\,\frac{2v_0^2\cos^2\!\alpha}{g} \quad\Rightarrow\quad R^2 - \frac{2v_0^2 R}{g}\sin\alpha\cos\alpha + \frac{2v_0^2 h}{g}\cos^2\!\alpha \;=\; 0 \tag{3.3.4}$$

$$\Rightarrow\quad R \;=\; \frac{v_0^2}{g}\sin\alpha\cos\alpha \pm \sqrt{\frac{v_0^4}{g^2}\sin^2\!\alpha\cos^2\!\alpha - \frac{2v_0^2 h}{g}\cos^2\!\alpha}$$

$$=\; \frac{v_0^2}{g}\sin\alpha\cos\alpha\left(1 \pm \sqrt{1 - \frac{2gh}{v_0^2\sin^2\!\alpha}}\right) \tag{3.3.5}$$

$$=\; \frac{\dot{x}_0\dot{z}_0}{g}\left(1 \pm \sqrt{1 - \frac{2gh}{\dot{z}_0^2}}\right).$$

For $2gh \;<\; \dot{z}_0^2$ $(mgh < \frac{1}{2}m\dot{z}_0^2)$, $\exists$ two roots; R is the greater $(+)$;

$= \; \dot{z}_0^2$ $(mgh = \frac{1}{2}m\dot{z}_0^2)$, $h = \dot{z}_0^2/2g = H$, maximum height, $R = \dfrac{\dot{x}_0\dot{z}_0}{g}$;

$> \; \dot{z}_0^2$ $(mgh > \frac{1}{2}m\dot{z}_0^2)$, $\exists$ no real roots; target cannot be hit.

Note that for $h = 0$ (PHYS 1210),

$$R \;=\; R_0 \;=\; \frac{v_0^2}{g}\sin\alpha\cos\alpha\,(2) \;=\; \frac{v_0^2}{g}\sin 2\alpha \;=\; \frac{2\dot{x}_0\dot{z}_0}{g}.$$

b) At what α is R maximised, and what is $R_{\max}$?

Solution: To find $R_{\max}$, set $\dfrac{dR}{d\alpha} = 0$. Differentiating Eq. (3.3.5) looks hard. Instead, differentiate Eq. (3.3.4) *implicitly* and set $\dfrac{dR}{d\alpha} = 0$ where it occurs:

$$2R_{\max}\cancelto{0}{\frac{dR}{d\alpha}} - \frac{2v_0^2}{g}\cancelto{0}{\frac{dR}{d\alpha}}\sin\alpha\cos\alpha$$
$$-\frac{2v_0^2}{g}R_{\max}(\underbrace{\cos^2\alpha - \sin^2\alpha}_{\cos 2\alpha}) - \frac{2v_0^2}{g}h\,\underbrace{2\sin\alpha\cos\alpha}_{\sin 2\alpha} = 0$$

$$\Rightarrow \quad R_{\max} = -h\tan 2\alpha, \tag{3.3.6}$$

where $-$ sign $\Rightarrow$ $2\alpha > 90°$, $\alpha > 45°$. Substitute this into Eq. (3.3.4) to get:

$$h^{\cancel{2}}\tan^2 2\alpha + \frac{v_0^2\cancel{h}}{g}\big(\tan 2\alpha\, 2\sin\alpha\cos\alpha + 2\cos^2\alpha\big) = 0 \tag{3.3.7}$$

which we must solve for α.

Aside: Trig strategy:

- simplify expressions with different multiples of an angle by using identities such as: $\sin 2\alpha = 2\sin\alpha\cos\alpha$; $\cos 2\alpha + 1 = 2\cos^2\alpha$; *etc.*
- reduce all trig functions to sin and cos, then just a sin or a cos.
- avoid like the plague inverse trig functions!

Using above identities on Eq. (3.3.7), we get:

$$h\tan^2 2\alpha + \frac{v_0^2}{g}\big(\underbrace{\tan 2\alpha\sin 2\alpha + \cos 2\alpha + 1}_{\equiv\, f(2\alpha)}\big) = 0.$$

But,

$$f(2\alpha) = \frac{\sin^2 2\alpha}{\cos 2\alpha} + \cos 2\alpha + 1 = \frac{\sin^2 2\alpha + \cos^2 2\alpha + \cos 2\alpha}{\cos 2\alpha} = \frac{1+\cos 2\alpha}{\cos 2\alpha},$$

and we have,

$$h\frac{\sin^2 2\alpha}{\cos^{\not 2} 2\alpha} + \frac{v_0^2}{g}\frac{1+\cos 2\alpha}{\cancel{\cos 2\alpha}} = \frac{h}{\cos 2\alpha}\underbrace{(1-\cos^2 2\alpha)}_{(1-\cos 2\alpha)\cancel{(1+\cos 2\alpha)}} + \frac{v_0^2}{g}\cancel{(1+\cos 2\alpha)} = 0$$

$$\Rightarrow \quad h - h\cos 2\alpha + \frac{v_0^2}{g}\cos 2\alpha = 0 \quad \Rightarrow \quad \cos 2\alpha = \frac{gh}{gh - v_0^2}, \qquad (3.3.8)$$

giving angle for $R_{\max}$. Thus, from Eq. (3.3.6),

$$R_{\max} = -h\frac{\sin 2\alpha}{\cos 2\alpha} = -h\frac{\sqrt{1-\cos^2 2\alpha}}{\cos 2\alpha} = \cancel{h}\sqrt{1-\left(\frac{gh}{gh-v_0^2}\right)^2}\,\frac{v_0^2-gh}{g\cancel{h}}$$

$$= \frac{1}{g}\sqrt{(v_0^2-gh)^2 - g^2h^2} = \frac{1}{g}\sqrt{v_0^4 - 2v_0^2 gh}$$

$$\Rightarrow \quad \boxed{R_{\max} = \frac{v_0^2}{g}\sqrt{1-\frac{2gh}{v_0^2}}.}$$

Notes.

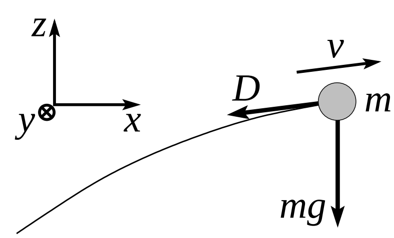
1. For $h = 0$, $\cos 2\alpha = 0 \Rightarrow \alpha = 45^\circ$ and $R_{\max} = v_0^2/g$ (PHYS 1201).

2. $R_{\max} \notin \mathbb{R}$ when $2gh > v_0^2$. Why does this make sense? □

End material for midterm 2.

3.3.2 With air resistance

Define $\vec{D} = -m\gamma\vec{v}$ (N) as *linear air drag*, where γ is *coefficient of linear air drag* (s^{-1}), and $\vec{v} = \dot{x}\hat{\imath} + \dot{z}\hat{k}$ is projectile velocity.



$$\Rightarrow \quad \sum \vec{F} = \cancel{m}\vec{g} - \cancel{m}\gamma\vec{v} = \cancel{m}\vec{a}.$$

$$\Rightarrow \quad \begin{cases} x/ \quad -\gamma\dot{x} = \ddot{x} \quad \Rightarrow \quad \dot{v}_x + \gamma v_x = 0 & (3.3.9) \\ z/ \quad -g - \gamma\dot{z} = \ddot{z} \quad \Rightarrow \quad \dot{v}_z + \gamma v_z = -g & (3.3.10) \end{cases}$$

- $\vec{a}$ is not constant, thus cannot use kinematics.

- $\vec{F}$ is separable, but not conservative (because $\vec{F}$ depends upon $\vec{v}$).

Eq. (3.3.9) is a separable[15], 1st order ODE (§1.6):

$$\frac{dv_x}{dt} = -\gamma v_x \quad \Rightarrow \quad \int_{v_{0x}}^{v_x} \frac{dv_x'}{v_x'} = -\gamma \int_0^t dt' \quad \Rightarrow \quad \ln\left(\frac{v_x}{v_{0x}}\right) = -\gamma t$$
$$\Rightarrow \quad v_x(t) = v_{0x} e^{-\gamma t},$$

where $v_{0x} = v_0 \cos\alpha$. Now, since $dx = v_x dt$, we have,

$$x(t) = \int_0^x dx' = \int_0^t v_x(t')\,dt' = v_{0x} \int_0^t e^{-\gamma t'} dt' = -\frac{v_{0x}}{\gamma}(e^{-\gamma t} - 1),$$

setting $x = 0$ at $t = 0$. Thus,

$$\boxed{x(t) = \frac{v_{0x}}{\gamma}(1 - e^{-\gamma t}).} \qquad (3.3.11)$$

Eq. (3.3.10) is also separable:

$$\frac{dv_z}{dt} = -\gamma v_z - g = -\gamma\left(v_z + \frac{g}{\gamma}\right) \quad \Rightarrow \quad \frac{dv_z}{v_z + g/\gamma} = -\gamma dt$$

$$\Rightarrow \quad \int_{v_{0z}}^{v_z} \frac{dv_z'}{v_z' + g/\gamma} = \ln\left(v_z' + \frac{g}{\gamma}\right)\bigg|_{v_{0z}}^{v_z} = \ln\left(\frac{v_z + g/\gamma}{v_{0z} + g/\gamma}\right)$$
$$= -\gamma \int_0^t dt' = -\gamma t$$

$$\Rightarrow \quad \frac{v_z + g/\gamma}{v_{0z} + g/\gamma} = e^{-\gamma t} \quad \Rightarrow \quad v_z = \left(v_{0z} + \frac{g}{\gamma}\right) e^{-\gamma t} - \frac{g}{\gamma}.$$

[15] Note the two different uses of the word *separable* here and in the line above.

$$\Rightarrow \quad z(t) = \int_0^t v_z(t')\,dt' = \int_0^t \left[\left(v_{0z} + \frac{g}{\gamma}\right) e^{-\gamma t'} - \frac{g}{\gamma}\right] dt'$$

$$\Rightarrow \quad \boxed{z(t) = \frac{1}{\gamma}\left(v_{0z} + \frac{g}{\gamma}\right)\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma},} \tag{3.3.12}$$

setting $z = 0$ at $t = 0$.

Combine Eq. (3.3.11), (3.3.12) to get one vector equation:

$$\vec{r}(t) = x(t)\,\hat{\imath} + z(t)\,\hat{k} = \vec{v}_0 \frac{1 - e^{-\gamma t}}{\gamma} + \frac{g\hat{k}}{\gamma}\left(\frac{1 - e^{-\gamma t}}{\gamma} - t\right)$$

$$\Rightarrow \quad \vec{r}(t) = \frac{1}{\gamma}\left[\left(\vec{v}_0 + \frac{g}{\gamma}\hat{k}\right)\left(1 - e^{-\gamma t}\right) - gt\hat{k}\right]. \tag{3.3.13}$$

So how does Eq. (3.3.13) reduce to Eq. (3.3.2),

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t - \frac{1}{2} g t^2 \hat{k}.$$

for $\gamma = 0$? Perform a *Maclaurin expansion* on $e^{-\gamma t}$:

$$1 - e^{-\gamma t} = 1 - \left(1 - \gamma t + \frac{(\gamma t)^2}{2!} + \dots\right) = \gamma t - \frac{\gamma^2 t^2}{2} + \dots,$$

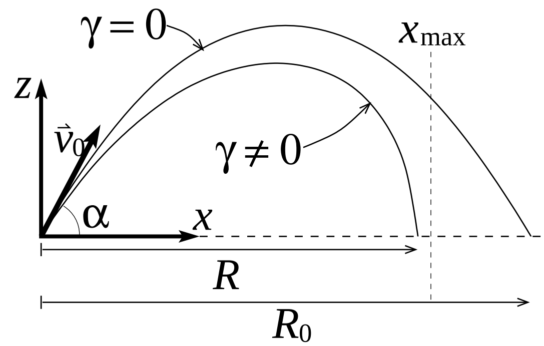
for $\gamma \to 0$. Thus, Eq. (3.3.13) becomes:

$$\begin{aligned}
\vec{r}(t) &= \frac{1}{\gamma}\left[\left(\vec{v}_0 + \frac{g}{\gamma}\hat{k}\right)\left(\gamma t - \frac{\gamma^2 t^2}{2} + \dots\right) - gt\hat{k}\right] \\
&= \frac{1}{\gamma}\left(\vec{v}_0 \gamma t - \vec{v}_0 \frac{\gamma^2 t^2}{2} + \cancel{gt\hat{k}} - \gamma \frac{gt^2}{2}\hat{k} + \dots - \cancel{gt\hat{k}}\right) \\
&= \vec{v}_0 t - \gamma \frac{\vec{v}_0 t^2}{2} - \frac{gt^2}{2}\hat{k} + \mathcal{O}(\gamma^2) \;\to\; \vec{v}_0 t - \tfrac{1}{2} g t^2 \hat{k} \;\text{ as } \gamma \to 0,
\end{aligned}$$

which is Eq. (3.3.2) for $\vec{r}_0 = 0$.

Trajectory, $z(x)$ (Assignment 7), found by eliminating t between Eq. (3.3.11), (3.3.12):

$$z(x) = \frac{x}{v_{0x}}\left(v_{0z} + \frac{g}{\gamma}\right) + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x}{v_{0x}}\right). \tag{3.3.14}$$



Unlike Eq. (3.3.3), Eq. (3.3.14) isn't a parabola. If projectile aloft long enough ($t \to \infty$), $x \to x_{\max} = v_{0x}/\gamma$ (Eq. 3.3.11).

For $t < \infty$, the range, $R < x_{\max}$. To find R, set $x = R$, $z(R) = 0$ in Eq. (3.3.14):

$$\frac{R}{v_{0x}}\left(v_{0z} + \frac{g}{\gamma}\right) + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma R}{v_{0x}}\right) \;=\; 0, \tag{3.3.15}$$

and do a Maclaurin expansion on the logarithm, namely Eq. (2.1.2):

$$\ln(1-u) \;=\; -u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \ldots,$$

convergent for $|u| < 1$. Thus, Eq. (3.3.15) becomes:

$$\begin{aligned}
&\frac{R}{v_{0x}}\left(v_{0z} + \frac{g}{\gamma}\right) + \frac{g}{\gamma^2}\left(-\frac{\gamma R}{v_{0x}} - \frac{(\gamma R)^2}{2v_{0x}^2} - \frac{(\gamma R)^3}{3v_{0x}^3} - \ldots\right) \\
&\quad = \frac{v_{0z}}{v_{0x}}R + \cancel{\frac{Rg}{v_{0x}\gamma}} - \cancel{\frac{gR}{\gamma v_{0x}}} - \frac{gR^2}{2v_{0x}^2} - \frac{g\gamma R^3}{3v_{0x}^3} - \ldots \;=\; 0 \\
&\times\frac{v_{0x}}{R} \qquad \Rightarrow \qquad v_{0z} - \frac{gR}{2v_{0x}} - \frac{g\gamma R^2}{3v_{0x}^2} - \mathcal{O}(\gamma^2) \;=\; 0,
\end{aligned}$$

a quadratic in R.

Remaining details (use quadratic formula to solve for R, then do a binomial expansion on radical) left to Assignment 7, where you'll find:

$$\Rightarrow \quad \boxed{R \;=\; \underbrace{\frac{v_0^2}{g}\sin 2\alpha}_{R_0}\left(1 - \frac{4v_0\gamma}{3g}\sin\alpha + \mathcal{O}(\gamma^2)\right),} \tag{3.3.16}$$

since $v_{0x} = v_0\cos\alpha$ and $v_{0z} = v_0\sin\alpha$.

Tutorial 6.3

Problem 4 (FC 4.14) A projectile of mass m moves within the z-x plane, with z the vertical direction.

a) Write down the components of the differential equation of motion for the projectile when the air resistance is given by $\vec{F}_{\text{drag}} = -c_2 v^2 \hat{v}$, where c_2 is a constant, v is the speed of the projectile, and $\hat{v}$ is a unit vector in the direction of the velocity (thus, $\vec{v} = v\hat{v}$).

b) Are the equations separated? Explain.

c) Show that the x component of the velocity is given by: $\dot{x} = \dot{x}_0 e^{-\gamma s}$, where $\gamma = c_2/m$ and s is the distance the projectile has moved along its path.

Lesson 17

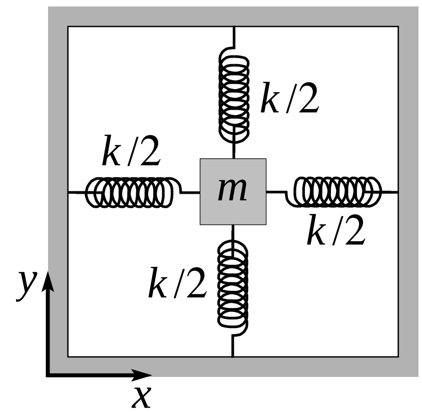
In this lesson, we discuss *multi-dimensional simple harmonic oscillators* (neither damped nor driven) as another example of motion in 3-D. In particular, we:

1. distinguish between *isotropic* and *non-isotropic* oscillators;
2. determine the path taken by an isotropic oscillator using *conics*;
3. discuss the qualitative differences observed in a non-isotropic oscillator; and
4. define a *commensurate non-isotropic oscillator*, and show how the *Lissajous figures* come about.

3.4 Multi-dimensional oscillators (FC §4.4)

Multi-dimensional oscillator: "springs" attached in more than 1-D.

Isotropic oscillator: a multidimensional oscillator with same spring constant, $k/2$, in each direction[16].



$$\vec{F} = -k\vec{r} = m\frac{d^2\vec{r}}{dt^2} \quad \Rightarrow \quad \begin{cases} m\ddot{x} = -kx; \\ m\ddot{y} = -ky; \\ m\ddot{z} = -kz. \end{cases}$$

$\vec{F}$ is both separable and conservative.

Demo: 2-D oscillator

Start with 2-D ($z = 0$). Then,

$$x(t) = A\cos(\omega_0 t + \alpha); \qquad y(t) = B\cos(\omega_0 t + \beta),$$

where $\omega_0 = \sqrt{k/m}$. A, B are amplitudes, α, β are phases; all constants of integration set from initial conditions.

To find path, $y(x)$, traced out by m, set $\beta = \alpha + \Delta$. Then,

$$y = B\cos(\omega_0 t + \alpha + \Delta) = B\big(\cos(\omega_0 t + \alpha)\cos\Delta - \sin(\omega_0 t + \alpha)\sin\Delta\big)$$

$$= B\left(\frac{x}{A}\cos\Delta - \sqrt{1 - \frac{x^2}{A^2}}\sin\Delta\right)$$

$$\Rightarrow \quad \frac{x}{A}\cos\Delta - \frac{y}{B} = \sqrt{1 - \frac{x^2}{A^2}}\sin\Delta$$

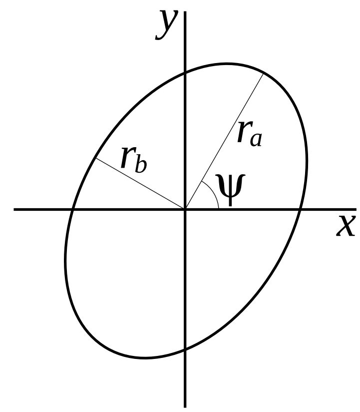
$$\Rightarrow \quad \frac{x^2}{A^2}\cos^2\Delta - 2\frac{xy}{AB}\cos\Delta + \frac{y^2}{B^2} = \sin^2\Delta - \frac{x^2}{A^2}\sin^2\Delta$$

$$\Rightarrow \quad \frac{1}{A^2}x^2 - \frac{2\cos\Delta}{AB}xy + \frac{1}{B^2}y^2 = \sin^2\Delta.$$

[16] F&C implicitly assume each spring has spring constant $k/2$ to add up to k in each direction.

A *conic*, given by: $ax^2 + bxy + cy^2 + dx + ey = f$, with the "discriminant" $D = b^2 - 4ac$, describes (*show handout* `conics.pdf`):

$$\begin{aligned} \text{an ellipse, if } D &< 0 \quad (\text{circle, if } A = B); \\ \text{a parabola, if } D &= 0; \\ \text{an hyperbola, if } D &> 0. \end{aligned}$$



In this case,

$$D = \frac{4\cos^2\Delta}{A^2B^2} - \frac{4}{A^2B^2} = -\frac{4\sin^2\Delta}{A^2B^2} < 0 \;\Rightarrow\; \text{ellipse.}$$

Further, an ellipse of form $ax^2 + bxy + cy^2 = f$ has semi-major axis, r_a, rotated by angle ψ relative to $+x$-axis, where (problem 4.18, ed. 7[17]),

$$\tan 2\psi \;=\; \frac{b}{a-c} \;=\; -\frac{2\cos\Delta}{AB}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)^{-1} \;=\; \frac{2AB\cos\Delta}{A^2 - B^2}.$$

If $\Delta = \pi/2$ (x, y motion out of phase by 90°), $\tan 2\psi = 0 \;\Rightarrow\; \psi = 0$ and ellipse is aligned with x-y axes.

If $\Delta = 0$ (x, y motion in phase) or π (out of phase by 180°), $\cos\Delta = \pm 1$ and $\sin\Delta = 0$,

$$\Rightarrow \quad \frac{x^2}{A^2} \mp 2\frac{xy}{AB} + \frac{y^2}{B^2} \;=\; \left(\frac{x}{A} \mp \frac{y}{B}\right)^2 \;=\; 0$$

$$\Rightarrow \quad y \;=\; \pm\frac{B}{A}x \quad (+ \text{ for } \Delta = 0, \; - \text{ for } \Delta = \pi),$$

and path is a straight line.

It turns out 3-D is same as 2-D. We can always write:

$$x(t) \;=\; A\cos(\omega_0 t + \alpha); \quad y(t) \;=\; B\cos(\omega_0 t + \beta); \quad z(t) \;=\; C\cos(\omega_0 t + \gamma),$$

[17] Assignment 7 in even years.

$$\text{as}: \quad \left.\begin{aligned} x &= D_x \cos\omega_0 t + E_x \sin\omega_0 t \\ y &= D_y \cos\omega_0 t + E_y \sin\omega_0 t \\ z &= D_z \cos\omega_0 t + E_z \sin\omega_0 t \end{aligned}\right\} \quad \Rightarrow \quad \vec{r} = \vec{D}\cos\omega_0 t + \vec{E}\sin\omega_0 t.$$

Thus, $\vec{r}$ (which describes path taken by m under influence of springs in 3-D) is restricted to 2-D plane described by vectors $\vec{D}$, $\vec{E}$.

Non-isotropic oscillator: a multidimensional oscillator where $k_x \neq k_y \neq k_z$.

$$\Rightarrow \quad \left.\begin{aligned} m\ddot{x} &= -k_x x, \\ m\ddot{y} &= -k_y y, \\ m\ddot{z} &= -k_z z, \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} x(t) &= A\cos(\omega_x t + \alpha); \\ y(t) &= B\cos(\omega_y t + \beta); \\ z(t) &= C\cos(\omega_z t + \gamma), \end{aligned}\right.$$

where $\omega_i = \sqrt{k_i/m}$, $i = x, y, z$.

In general, m follows a semi-chaotic, non-planar path. All points within a box $(2A, 2B, 2C)$ centred on the origin are visited.

ω_x, ω_y, and ω_z are said to be *commensurate* if:

$$\frac{\omega_x}{n_x} = \frac{\omega_y}{n_y} = \frac{\omega_z}{n_z},$$

where $n_\xi \in \mathbb{Z}$ with no common factors.

In this case, m returns to its initial conditions (position and velocity) after time $T = 2\pi n_x/\omega_x = 2\pi n_y/\omega_y = 2\pi n_z/\omega_z$, and retraces its path.

Path for non-isotropic but commensurate oscillator is closed, and only a portion of box $(2A, 2B, 2C)$ is visited.

Example 3.8. Find path of a particle moving within potential:

$$U(x, y) = \tfrac{1}{2}k(x^2 + 4y^2),$$

with $x = a$, $y = 0$, $\dot{x} = 0$, and $\dot{y} = v_0$ at $t = 0$.

Solution: Starting with Eq. (3.2.3),

$$\vec{F} = -\nabla U = -kx\,\hat{\imath} - 4ky\,\hat{\jmath} = m\ddot{\vec{r}} \quad \Rightarrow \quad \begin{cases} m\ddot{x} = -kx \\ m\ddot{y} = -4ky \end{cases}$$

$$\Rightarrow \begin{cases} x(t) = A_x \cos\omega_x t + B_x \sin\omega_x t \\ y(t) = A_y \cos\omega_y t + B_y \sin\omega_y t \end{cases}$$

$$\text{and} \begin{cases} v_x(t) = -A_x\omega_x \sin\omega_x t + B_x\omega_x \cos\omega_x t \\ v_y(t) = -A_y\omega_y \sin\omega_y t + B_y\omega_y \cos\omega_y t \end{cases}$$

where $\omega_x = \sqrt{k/m} \equiv \omega$ and $\omega_y = \sqrt{4k/m} = 2\omega$.

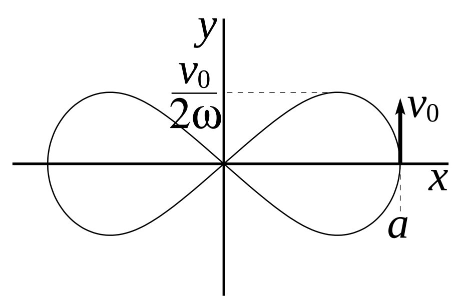
Applying initial conditions, we have:

$$x(0) = A_x = a; \quad y(0) = A_y = 0;$$

$$v_x(0) = B_x\omega = 0; \quad v_y(0) = B_y 2\omega = v_0,$$

whence,

$$\left.\begin{aligned} x(t) &= a\cos\omega t \\ y(t) &= \frac{v_0}{2\omega}\sin 2\omega t \end{aligned}\right\} \quad \Rightarrow \quad \vec{r}(t) = \left(a\cos\omega t,\, \frac{v_0}{2\omega}\sin 2\omega t,\, 0\right).$$



- for every half cycle traversed in x, m traverses a full cycle in y
- $\Rightarrow$ a "figure-8" parameterised by ωt
- path closed because frequencies are commensurate: $\dfrac{\omega_x}{1} = \dfrac{\omega_y}{2} = \omega$.

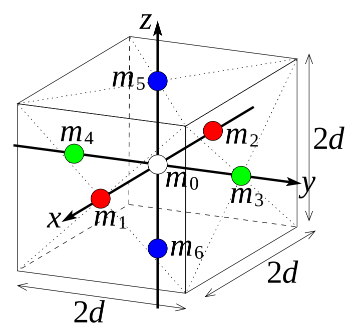
Such closed paths are *Lissajous* (Bowditch) *figures*, named for Jules Lissajous (1857), Nathanial Bowditch (1815) who first used them to study harmonics.

Tutorial 7.1

Problem 1 (FC 4.19) The potential energy of interaction between any two atoms in a simple cubic lattice has the form $U(r) \sim cr^{-\alpha}$, where c and α are constants, and where r is the distance between the atoms. Show that the total energy of interaction of a given atom with its six nearest neighbours is approximately the potential of a 3-D harmonic oscillator, namely:

$$U(x, y, z) \;\sim\; A + B(x^2 + y^2 + z^2), \tag{1}$$

where A and B are constants, and where (x, y, z) is the displacement of the given atom from its equilibrium position.



Hint: As shown in the diagram, place the atom of interest, m_0, at the origin, its six nearest neighbours ($m_1 \to m_6$) at $(\pm d, 0, 0)$, $(0, \pm d, 0)$, and $(0, 0, \pm d)$, and then displace m_0 by an arbitrary displacement (x, y, z) such that $x^2 + y^2 + z^2 \ll d^2$. Then,

$$U(x, y, z) \;=\; c\sum_{i=1}^{6} r_i^{-\alpha}, \tag{2}$$

where $r_1 = \sqrt{(d-x)^2 + y^2 + z^2}$, and *etc.* for $r_2, \ldots, r_6$. The approximation formulæ in Appendix D of the text may help.

Lesson 18

As a final example of motion of a single particle in 3-D, we:

1. introduce the electromagnetic *Coulomb-Lorentz force*; and
2. find the trajectories of a point charge moving within a,
    - pure electric field,
    - pure magnetic field.

This second half of this lesson introduces accelerating reference frames by considering *translational* (non-rotating) motion only. Here we look at the differences between observing motion from an *inertial* frame of reference, and from one that is accelerating in a straight line.

3.5 Electromagnetic forces (FC §4.5)

The combined Coulomb-Lorentz force law is:

$$\vec{F} \;=\; q(\vec{E} + \vec{v} \times \vec{B}) \;=\; m\ddot{\vec{r}},$$

which, in general, is neither separable nor conservative.

Example 3.9. Find trajectory of charge q moving in $\vec{E} = -E_0\hat{k}$, $\vec{B} = 0$.

Solution: $\vec{F} = -qE_0\hat{k} = m\ddot{z}\hat{k} \;\Rightarrow\; \ddot{z} = -\dfrac{qE_0}{m}$, identical in form as $\ddot{z} = -g$ for projectiles near earth's surface.

Thus, if $\vec{r}(0) = 0$ and $\vec{v}(0) = v_0$ at angle α relative to $\hat{\imath}$, trajectory is:

$$z(x) \;=\; x\tan\alpha - \frac{qE_0x^2}{2mv_0^2\cos^2\alpha},$$

same as Eq. (3.3.3).

If $\vec{E}$ arises from static charges only (*i.e.*, $\partial_t\vec{B} = 0$), then Faraday's Law $\Rightarrow\; \nabla \times \vec{E} = 0$ and $\vec{F}_E = q\vec{E}$ is conservative (also like gravity). □

Example 3.10. Find trajectory and speed of q moving in $\vec{B} = B_0\hat{k}$, $\vec{E} = 0$.

Solution: With $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$,

$$\vec{F} \;=\; qB_0\vec{v} \times \hat{k} \;=\; qB_0(-\dot{x}\,\hat{\jmath} + \dot{y}\,\hat{\imath}) \;=\; m(\ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath} + \ddot{z}\,\hat{k}). \tag{3.5.1}$$

Equating components, we get:

$$\Rightarrow \quad m\ddot{x} \;=\; qB_0\dot{y}; \qquad m\ddot{y} \;=\; -qB_0\dot{x}; \qquad \ddot{z} \;=\; 0, \tag{3.5.2}$$

where ODEs are coupled (non-separable). Integrate once to get:

$$m\dot{x} \;=\; qB_0y + c_1; \qquad m\dot{y} \;=\; -qB_0x + c_2; \qquad \dot{z} \;=\; \dot{z}_0. \tag{3.5.3}$$

Substitute second of Eq. (3.5.3) into first of Eq. (3.5.2) to get:

$$m\ddot{x} \;=\; \frac{qB_0}{m}(-qB_0x + c_2) \quad\Rightarrow\quad \ddot{x} \;=\; -\left(\frac{qB_0}{m}\right)^2 x + \frac{qB_0c_2}{m^2}.$$

Define $\omega = \dfrac{qB_0}{m}$ (s^{-1}) and $a = \dfrac{c_2}{qB_0} = \dfrac{c_2}{m\omega}$ (m). Then,

$$\ddot{x} + \omega^2 x \;=\; \omega^2 a,$$

where homogeneous equation (RHS = 0) is the SHO! Thus, solution is:

$$x(t) \;=\; x_{\mathrm{h}}(t) + x_{\mathrm{p}} \;=\; A\cos(\omega t + \delta) + a, \tag{3.5.4}$$

where A and δ are constants of integration.

Next, differentiate Eq. (3.5.4) to get:

$$\dot{x}(t) \;=\; -A\omega\sin(\omega t + \delta), \tag{3.5.5}$$

and substitute this into first of Eq. (3.5.3) to get,

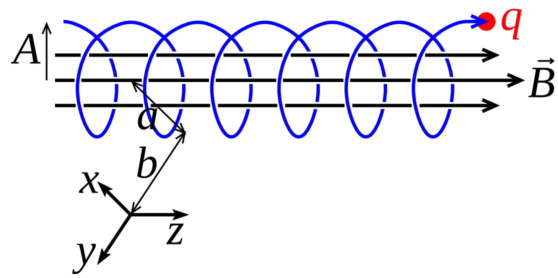
$$-mA\omega\sin(\omega t + \delta) \;=\; -qB_0A\sin(\omega t + \delta) \;=\; qB_0y + c_1$$

$$\Rightarrow \quad y(t) \;=\; b - A\sin(\omega t + \delta) \quad \text{and} \quad \dot{y}(t) \;=\; -A\omega\cos(\omega t + \delta), \tag{3.5.6}$$

where $b = -\dfrac{c_1}{qB_0}$.

For trajectory, eliminate t between Eq. (3.5.4) and first of Eq. (3.5.6). Most efficient way:

$$\begin{array}{rcl} \text{Eq. (3.5.4)} & \Rightarrow & A^2\cos^2(\omega t + \delta) \;=\; (x-a)^2 \\ \text{Eq. (3.5.6)} & \Rightarrow & A^2\sin^2(\omega t + \delta) \;=\; (b-y)^2 \\ \hline \text{add} & \Rightarrow & A^2 \;=\; (x-a)^2 + (y-b)^2, \end{array}$$



a circle of radius A centred at (a, b, z). Since $\dot{z}$ = constant, path is an open helix with axis $\parallel \hat{k} \parallel \vec{B}$.

As for $\vec{v}$,

$$v^2 = \underbrace{\dot{x}^2 + \dot{y}^2}_{v_\perp^2} + \underbrace{\dot{z}^2}_{v_\parallel^2} = A^2\omega^2 + \dot{z}_0^2 \quad \Rightarrow \quad v_\perp = A\omega = \frac{AqB_0}{m} = \text{constant},$$

using third of Eq. (3.5.3), Eq. (3.5.5) and second of Eq. (3.5.6). Thus, *radius of gyration* is:

$$A = \frac{mv_\perp}{qB_0},$$

and charge q moves at constant speed v along a helix with radius given by the radius of gyration, A, and whose axis is parallel to $\vec{B}$. □

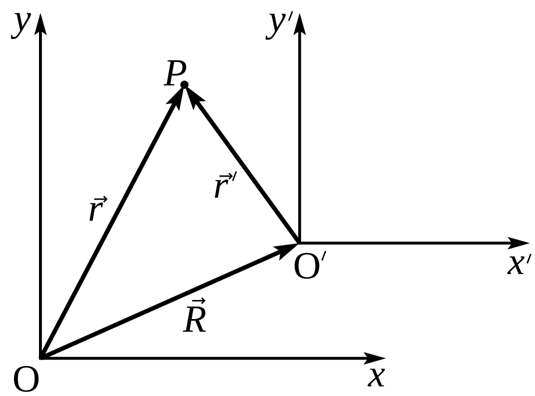
Part IV: Accelerating Reference Frames

Reading assignment: Chapter 5

4.1 Translational Motion (FC §5.1)

Let O be an *inertial frame of reference* in which Newton's Laws are valid:

$$\vec{F} = \sum_i \vec{F}_i = m\vec{a} = m\ddot{\vec{r}}. \qquad (4.1.1)$$



Let O′ be an accelerating frame;

$\vec{r}$ = position of P relative to O;

$\vec{r}'$ = position of P relative to O′;

$\vec{R}$ = position of O′ relative to O,

where,

$$\vec{r} = \vec{r}' + \vec{R} \quad \Rightarrow \quad \vec{v} = \vec{v}' + \vec{V} \quad \Rightarrow \quad \vec{a} = \vec{a}' + \vec{A}. \qquad (4.1.2)$$

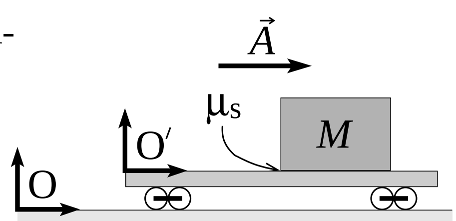
Substitute last of Eq. (4.1.2) into (4.1.1) to get:

$$\vec{F} = m\vec{a}' + m\vec{A} \quad \Rightarrow \quad \vec{F} - m\vec{A} \equiv \boxed{\vec{F}' = m\vec{a}',}$$

a variation of Newton's 2nd law where $\vec{F}'$ is:

- $\vec{F}$; *real forces* from physical object or field (*e.g.*, T, kx, mg, *etc.*), plus
- *inertial or fictitious forces*; caused by acceleration of reference frame (*e.g.*, $-m\vec{A}$; always $\propto$ mass)

Example 4.1. From frames O′ and O, find minimum A that causes M to slip.

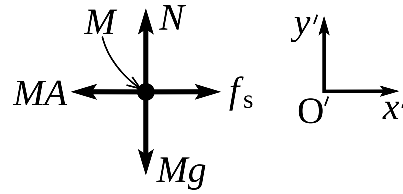


Solution: In O′, $a' = 0$. Thus,

$$y/\quad N - Mg = 0 \quad \Rightarrow \quad N = Mg;$$

$$x/\quad -MA + f_{\mathrm{s}} = 0$$

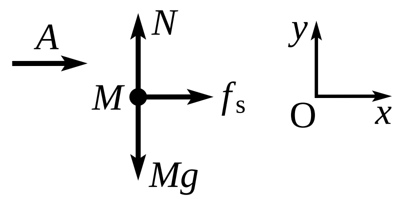
$$\Rightarrow \quad A = \frac{f_{\mathrm{s}}}{M} \leq \frac{\mu_{\mathrm{s}} N}{M} = \mu_{\mathrm{s}} g.$$



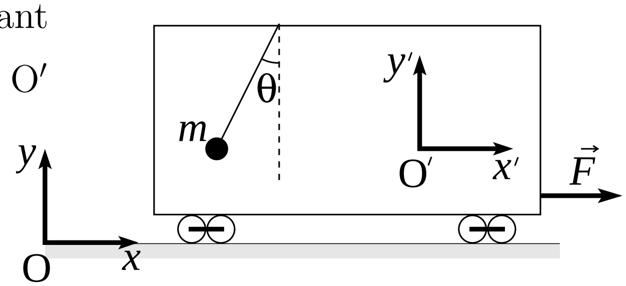
Thus, M slips once $A > \mu_{\mathrm{s}} g$. Next, in non-accelerating frame, O:

$$x/\quad f_{\mathrm{s}} = MA \leq \mu_{\mathrm{s}} N = \mu_{\mathrm{s}} Mg$$

$$\Rightarrow \quad A \leq \mu_{\mathrm{s}} g \quad \text{for no slipping.}$$

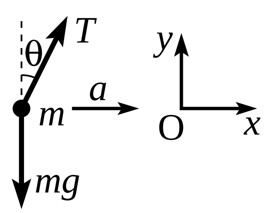


Example 4.2. Boxcar pulled with constant force F on a smooth track. How do O, O′ explain position of bob, m?



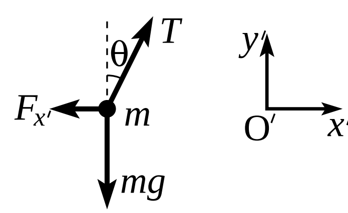
For O: $\sum \vec{F}_i = m\vec{a} \Rightarrow$

$$\left.\begin{aligned} x/\quad & T\sin\theta = ma \\ y/\quad & T\cos\theta - mg = 0 \end{aligned}\right\} \Rightarrow \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{a}{g}.$$



Thus, θ is angle needed for force $T\sin\theta$ to accelerate m at $a = g\tan\theta$.

For O′: $\sum \vec{F}_i' = m\vec{a}\,' = 0$ (m at rest relative to O′). Thus, some force, $F_{x'}$, must be pulling m to the left.



$$\left.\begin{aligned} x'/\quad & T\sin\theta - F_{x'} = 0 \\ y'/\quad & T\cos\theta - mg = 0 \end{aligned}\right\} \Rightarrow \frac{T\sin\theta}{T\cos\theta} = \frac{F_{x'}}{mg}$$

$$= \tan\theta \quad \Rightarrow \quad F_{x'} = mg\tan\theta.$$

Acting by itself, $F_{x'} = ma' \Rightarrow a' = g\tan\theta$ to left and a' independent of m.

- accelerations imparted by all inertial forces are independent of m.
- for all real forces *except gravity*, $a \propto \dfrac{1}{m}$. *e.g.*, electric force qE, $a = \dfrac{qE}{m}$.

Tutorial 7.2

Problem 2 Show that despite depending upon velocity, the Lorentz force as given by Eq. (3.5.1) is conservative. That is, show that $\nabla \times \vec{F} = 0$.

Tutorial 8.1

Problem 1 (FC 5.1) A physics student who weighs 640 N ($\sim$140 lb.) stands on a bathroom spring scale in a moving elevator. What is the weight indicated on the scale if the elevator is:

a) accelerating upward with acceleration $g/4$;

b) accelerating downward with acceleration $g/4$;

c) moving upward at constant speed;

d) moving downward at constant speed?

Problem 2 (FC 5.5) A truck driving on a level road suddenly slows with an acceleration $g/2$ causing a box of mass m in the rear to slide forward. If $\mu_\mathrm{k} = 1/3$ between the box and floor, find the acceleration of the box a) relative to the truck, and b) relative to the road.

Lesson 19

In this lesson, we include rotation in an accelerating frame of reference. By carefully considering the kinematical quantities ($\vec{r}$, $\vec{v}$, $\vec{a}$) in the rotating system, we uncover *three* new types of inertial acceleration:

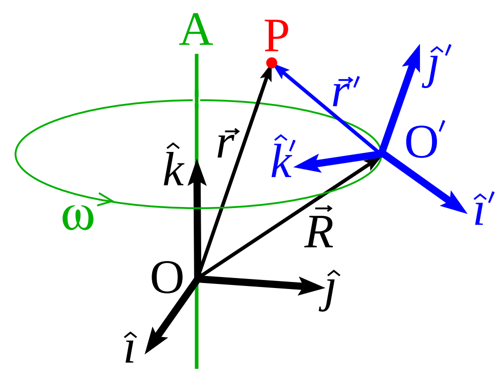
1. *transverse* acceleration, associated with the rate of change of rotation;
2. *centrifugal* acceleration, associated with rotation and displacement; and
3. *Coriolis* acceleration, associated with rotation and velocity.

We finish with a detailed example of how to assess both real and inertial accelerations using a "*hunting and gathering*" algorithm.

4.2 Rotational motion (FC §5.2)

Consider origins of two Cartesian coordinate systems, O (black) and O′ (blue) displaced by $\vec{R}$.

- O ($\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$) is inertial;
- O′ ($\hat{\imath}'$, $\hat{\jmath}'$, and $\hat{k}'$) is accelerating:
  - rotates about axis A[18];
  - $\ddot{\vec{R}}$ may be non-zero.



Without loss of generality, align $\hat{k}$ with A. Then, if $\vec{r}$ and $\vec{r}'$ are displacements of P relative to O and O′,

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \vec{r}' + \vec{R} = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}' + \vec{R}$$

$$\Rightarrow \quad \vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} \quad (\text{since } \dot{\hat{\imath}} = 0, \textit{ etc.})$$

$$= \frac{d\vec{r}'}{dt} + \dot{\vec{R}} = \underbrace{\dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}'}_{\vec{v}'} + x'\dot{\hat{\imath}}' + y'\dot{\hat{\jmath}}' + z'\dot{\hat{k}}' + \vec{V}, \qquad (4.2.1)$$

where: $\vec{V} = \dot{\vec{R}} =$ relative velocity between O and O′;

$\vec{v}' =$ velocity of P relative to O′[19].

and since $\dot{\hat{\imath}}' \neq 0$, *etc.* Thus, Eq. (4.2.1) $\Rightarrow$

$$\vec{v} = \vec{v}' + x'\dot{\hat{\imath}}' + y'\dot{\hat{\jmath}}' + z'\dot{\hat{k}}' + \vec{V}, \qquad (4.2.2)$$

and we seek expressions for $\dot{\hat{\imath}}'$, $\dot{\hat{\jmath}}'$, and $\dot{\hat{k}}'$.

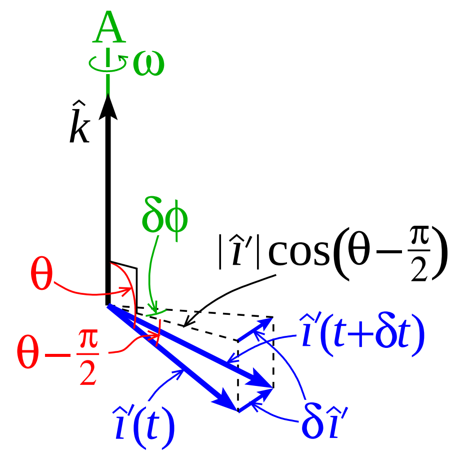
[18] Rotation is in a "tidally-locked" fashion where observer at O′ always faces rotation axis, A.

[19] $\vec{v}'$ doesn't include time derivatives of $\hat{\imath}'$, *etc.*, just as your velocity due east along a highway doesn't include how the unit vector pointing "due east" changes as the earth rotates.

As $\hat{\imath}'$ rotates, we see from the diagram,

$$|\delta\hat{\imath}'| = |\hat{\imath}'|\cos\left(\theta - \tfrac{\pi}{2}\right)\overset{\omega\delta t}{\cancel{\delta\phi}} = (1)(\omega)\sin\theta\,\delta t$$

where θ is angle between $\hat{\imath}'$ and $\vec{\omega}$ (A).



Now, $\delta\hat{\imath}' \perp \vec{\omega}$ and, for $|\delta\hat{\imath}'| \ll 1$, $\delta\hat{\imath}' \perp \hat{\imath}'$. Thus, $\delta\hat{\imath}' = \vec{\omega} \times \hat{\imath}'\delta t$ (right-hand rule),

$$\Rightarrow \quad \dot{\hat{\imath}}' = \vec{\omega} \times \hat{\imath}'.$$

Similarly, $\dot{\hat{\jmath}}' = \vec{\omega} \times \hat{\jmath}'$, $\dot{\hat{k}}' = \vec{\omega} \times \hat{k}'$ and Eq. (4.2.2) $\Rightarrow$

$$\vec{v} = \vec{v}' + x'\vec{\omega} \times \hat{\imath}' + y'\vec{\omega} \times \hat{\jmath}' + z'\vec{\omega} \times \hat{k}' + \vec{V} = \vec{v}' + \vec{\omega} \times \vec{r}' + \vec{V}. \qquad (4.2.3)$$

For acceleration, differentiate Eq. (4.2.3) to get,

$$\begin{aligned}
\vec{a} = \frac{d\vec{v}}{dt} &= \frac{d}{dt}\left(\vec{v}' + \vec{\omega} \times \vec{r}' + \vec{V}\right) \\
&= \frac{d}{dt}\left(\dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}'\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}' + \vec{\omega} \times \frac{d}{dt}\left(x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'\right) + \vec{A}, \\
&= \underbrace{\ddot{x}'\hat{\imath}' + \ddot{y}'\hat{\jmath}' + \ddot{z}'\hat{k}'}_{\vec{a}'} + \underbrace{\dot{x}'\dot{\hat{\imath}}' + \dot{y}'\dot{\hat{\jmath}}' + \dot{z}'\dot{\hat{k}}'}_{\vec{\omega} \times \vec{v}'} + \dot{\vec{\omega}} \times \vec{r}' \\
&\qquad + \vec{\omega} \times \Big(\underbrace{\dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}'}_{\vec{v}'} + \underbrace{x'\dot{\hat{\imath}}' + y'\dot{\hat{\jmath}}' + z'\dot{\hat{k}}'}_{\vec{\omega} \times \vec{r}'}\Big) + \vec{A}
\end{aligned}$$

$$\Rightarrow \quad \vec{a} = \vec{a}' + \dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}') + \vec{A}, \qquad (4.2.4)$$

where: $\vec{A} = \ddot{\vec{R}} =$ acceleration of O$'$ relative to O;

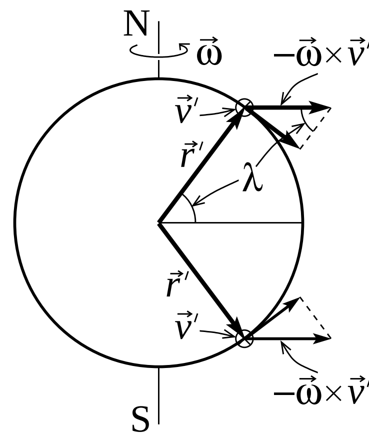
$\vec{a}' =$ acceleration of P relative to O$'$.

Solving Eq. (4.2.4) for $\vec{a}'$, we get,

$$\vec{a}' = \underbrace{\vec{a}}_{\text{real}} \underbrace{- \dot{\vec{\omega}} \times \vec{r}'}_{\text{transverse}} \underbrace{- 2\,\vec{\omega} \times \vec{v}'}_{\text{Coriolis}} \underbrace{- \vec{\omega} \times (\vec{\omega} \times \vec{r}')}_{\text{centrifugal}} \underbrace{- \vec{A}}_{\text{translational}}.$$

In addition to *real* acceleration, $\vec{a}$, four *inertial* accelerations arise:

1. *transverse*: $a_\perp = -\dot{\vec{\omega}} \times \vec{r}\,' \perp \vec{r}\,'$; $a_\perp$ *transverse* to displacement. Exists *only* when $\dot{\vec{\omega}} \neq 0$.



2. *Coriolis*: $-2\,\vec{\omega} \times \vec{v}\,' \perp \vec{v}\,'$ and rotation axis.

   On earth, if $\vec{v}\,'$ points E

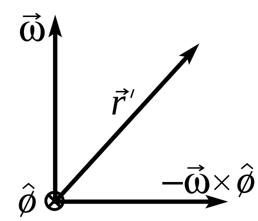
   $\Rightarrow$ $-\vec{\omega} \times \vec{v}\,'$ points away from rotation axis.

   $\Rightarrow$ ground component in N (S) points S (N).

3. *centrifugal*: $\vec{\omega} \times \vec{r}\,' \propto \hat{\phi}$ (points E on earth)

   $\Rightarrow$ $-\vec{\omega} \times \hat{\phi}$ points away from rotation axis.

   As viewed from O, term $\propto +\vec{\omega} \times \hat{\phi}$; points *toward* rotation axis—*centripetal* acceleration.



4. *translational*: $-\vec{A}$, as in §4.1.

Relationships between inertial and non-inertial kinematical quantities:

$$\boxed{\begin{aligned} \vec{r} &= \vec{r}\,' + \vec{R}; \\ \vec{v} &= \vec{v}\,' + \vec{\omega} \times \vec{r}\,' + \vec{V}; \\ \vec{a} &= \vec{a}\,' + \dot{\vec{\omega}} \times \vec{r}\,' + 2\,\vec{\omega} \times \vec{v}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') + \vec{A}. \end{aligned}} \tag{4.2.5}$$

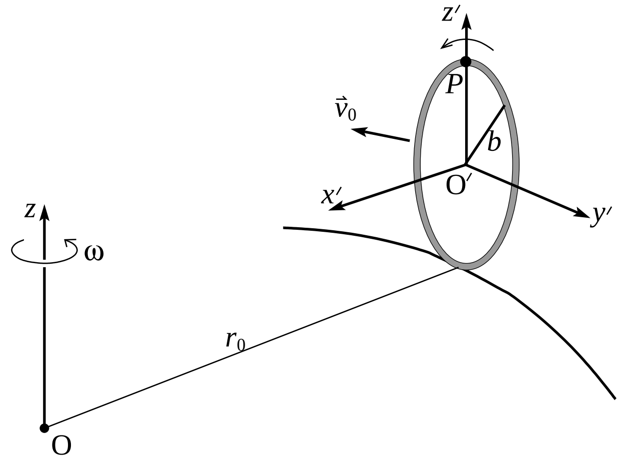
Example 4.3. A vertical bicycle wheel (radius b) moves around a circular track (radius r_0) at constant speed v_0. Find acceleration, $\vec{a}$, of point P (highest point on wheel) relative to an inertial observer, O, at centre of track.

With right choice of coordinate systems, this is a bookkeeping[20] problem.

[20]Fun fact: This and *bookkeeper* are the only words in the English language with three consecutive double letters!

Put O′ at wheel axle, let x' always point at O and z' remain upright.

Thus, O′ moves around track at centre of wheel without z' and y' rotating about axle.



Step 1: "Hunting and gathering".

$$
\begin{aligned}
&\text{acceleration of } P \text{ relative to O}': && \vec{a}\,' = \frac{v_0^2}{b}(-\hat{k}'); \\
&\text{angular velocity of O}' \text{ relative to O}: && \vec{\omega} = \frac{v_0}{r_0}\hat{k}' \quad (\text{not } P \text{ about O}'!); \\
&\text{angular acceleration of O}' \text{ relative to O}: && \dot{\vec{\omega}} = 0; \\
&\text{position of } P \text{ relative to O}': && \vec{r}\,' = b\hat{k}'; \\
&\text{velocity of } P \text{ relative to O}': && \vec{v}\,' = -v_0\hat{\jmath}\,'; \\
&\text{acceleration of O}' \text{ relative to O}: && \vec{A} = \frac{v_0^2}{r_0}\,\hat{\imath}\,'.
\end{aligned}
$$

Step 2: Do the cross products.

$$
\begin{aligned}
&\text{transverse term:} && \dot{\vec{\omega}} \times \vec{r}\,' = 0; \\
&\text{Coriolis term:} && 2\vec{\omega} \times \vec{v}\,' = \frac{2v_0^2}{r_0}\,\hat{k}' \times (-\hat{\jmath}\,') = \frac{2v_0^2}{r_0}\,\hat{\imath}\,'; \\
&\text{centrifugal term:} && \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') = \frac{v_0^2}{r_0^2} b\,\hat{k}' \times (\hat{k}' \times \hat{k}') = 0.
\end{aligned}
$$

Step 3: Assemble the accelerations (Eq. 4.2.5):

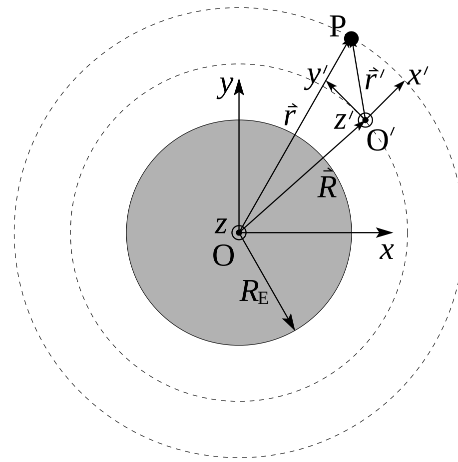
$$
\vec{a}_P = -\frac{v_0^2}{b}\hat{k}' + 0 + \frac{2v_0^2}{r_0}\,\hat{\imath}\,' + 0 + \frac{v_0^2}{r_0}\,\hat{\imath}\,' \quad \Rightarrow \quad \boxed{\vec{a}_P = \frac{3v_0^2}{r_0}\,\hat{\imath}\,' - \frac{v_0^2}{b}\hat{k}'.}
$$

Note we've expressed $\vec{a}$ (relative to O) in terms of unit vectors of O′. □

Tutorial 8.2

Problem 3 (FC 5.18) In the figure, objects P and O′ are "close-by" satellites in counter-clockwise circular orbits about the earth, where the z-axis is aligned with the north pole. Show that the acceleration of P relative to O′ is given by:

$$\ddot{x}' = 2\omega\dot{y}' + 3\omega^2 x'; \qquad \ddot{y}' = -2\omega\dot{x}'.$$



In the diagram, O is the origin of the *inertial* reference frame at the centre of the earth where the x-y axes do not rotate with the earth. The x'-y' axes are affixed to O′ and rotate about the earth with it.

Hint: To find the acceleration of P relative to O′, start with equation 5.2.14 (eds. 6 and 7), and assess each term as we've done in class. The only term that's zero is the $\dot{\omega}$ term. After doing some cross products, you should find:

$$\vec{a}' = (\omega^2 - \omega_{\mathrm{P}}^2)\vec{r} + 2\omega(\dot{y}'\hat{\imath}' - \dot{x}'\hat{\jmath}'),$$

where the first term on the RHS is proportional to $\vec{r}$ and not $\vec{r}'$. You'll also want Kepler's third law ($\omega^2 \propto r^{-3}$), and the fact that $r' \ll R$ will allow you to drop terms of order r'^2/R^2.

Lesson 20

This lesson considers the *dynamics* in a rotating frame of reference (essentially by multiplying the accelerations from the last lesson with the mass to get forces). In so doing, we derive *Coriolis' theorem* in which forces are classified as either *real* or *inertial*, where inertial forces come in four types:

1. *transverse force*, $\vec{F}_{\perp}$;
2. *Coriolis force*, $\vec{F}_{\text{Cor}}$;
3. *centrifugal force*, $\vec{F}_{\text{cent}}$; and
4. *translational force*, $\vec{F}_{\text{tr}}$.

We shall use the "*hunting and gathering*" method to solve a few problems in rotational dynamics.

We also consider the Earth as a rotating frame of reference where the "real" force of gravity and the "inertial" translational force are combined to give an "effective force of gravity", what we all think of as Earth's gravity.

4.3 Dynamics in an accelerating coordinate system (FC §5.3)

Multiply third of Eq. (4.2.5) by m to get,

$$\vec{F} = m\vec{a} = m\vec{a}' + m\dot{\vec{\omega}} \times \vec{r}' + 2m\vec{\omega} \times \vec{v}' + m\vec{\omega} \times (\vec{\omega} \times \vec{r}') + m\vec{A}$$

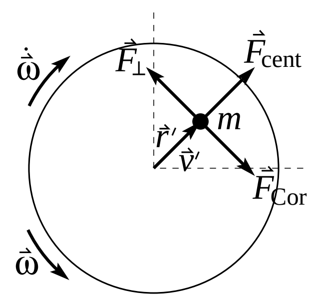
$$\Rightarrow \quad \vec{F}' = m\vec{a}' = \underset{\text{real}}{\underset{\uparrow}{\vec{F}}} \underbrace{- m\dot{\vec{\omega}} \times \vec{r}'}_{\text{transverse}} \underbrace{- 2m\vec{\omega} \times \vec{v}'}_{\text{Coriolis}} \underbrace{- m\vec{\omega} \times (\vec{\omega} \times \vec{r}')}_{\text{centrifugal}} \underbrace{- m\vec{A}}_{\text{translational}}$$

$$= \vec{F} + \vec{F}_{\perp} + \vec{F}_{\text{Cor}} + \vec{F}_{\text{cent}} + \vec{F}_{\text{tr}}, \tag{4.3.1}$$

Coriolis' theorem (1835): $\vec{F}'$ = sum of real forces and four *inertial forces.*

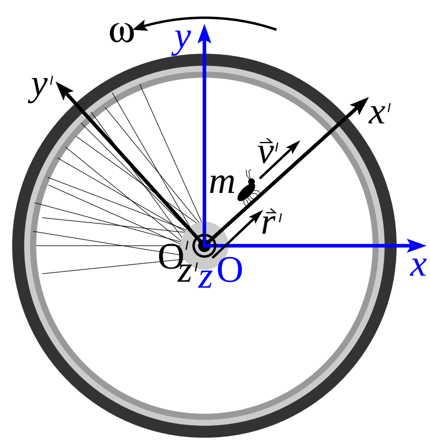
- inertial forces are all $\propto m$; arise entirely from acceleration of O′
- because $\vec{F}_g \propto m$, gravitational force treated as an inertial force in GR

Example 4.4. Consider a mass m moving outward radially on a rotating merry-go-round. Inertial forces in directions shown.



Is it easier to walk outward when merry-go-round is speeding up or slowing down? □

Example 4.5. A bug of mass m crawls at constant speed, v_0, along a spoke of a bicycle wheel spinning at constant ω in the horizontal plane.



a) Find "surface force", $\vec{S}(r')$, exerted by spoke on bug in the *horizontal plane of the wheel.*

Put origins of O′ and O on the axle and let x' always point along spoke with bug.

Step 1: "Hunting and gathering"

$$\begin{aligned}
&\text{acceleration of } m \text{ relative to O}': && \vec{a}\,' = 0;\\
&\text{angular velocity of O}' \text{ relative to O}: && \vec{\omega} = \omega\hat{k}'\\
&\text{angular acceleration of O}' \text{ relative to O}: && \dot{\vec{\omega}} = 0;\\
&\text{position of } m \text{ relative to O}': && \vec{r}\,' = r'\hat{\imath}\,';\\
&\text{velocity of } m \text{ relative to O}': && \vec{v}\,' = v_0\hat{\imath}\,';\\
&\text{acceleration of O}' \text{ relative to O}: && \vec{A} = 0 \quad \text{(O and O}' \text{ coincident)}
\end{aligned}$$

Step 2: Do the cross products.

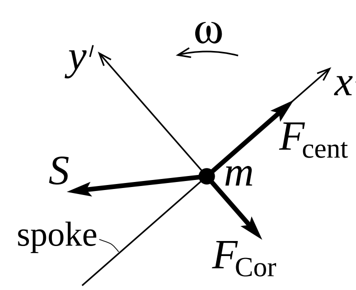
$$\begin{aligned}
&\text{transverse force:} && -m\dot{\vec{\omega}} \times \vec{r}\,' = 0;\\
&\text{Coriolis force:} && -2m\vec{\omega} \times \vec{v}\,' = -2m\omega v_0 \hat{k}' \times \hat{\imath}\,' = -2m\omega v_0 \hat{\jmath}\,';\\
&\text{centrifugal force:} && -m\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') = -m\omega^2 r'\,\hat{k}' \times (\hat{k}' \times \hat{\imath}\,') = m\omega^2 r'\hat{\imath}\,'.
\end{aligned}$$

Step 3: Assemble the forces (Eq. 4.3.1):

$$\vec{F}' = \vec{F} - 2m\omega v_0\hat{\jmath}\,' + m\omega^2 r'\hat{\imath}\,' = 0 \quad (\because \vec{a}\,' = 0)$$

$$\Rightarrow \quad \vec{F} = \text{all real forces in horizontal plane}$$

$$= \boxed{\vec{S} = -m\omega^2 r'\hat{\imath}\,' + 2m\omega v_0\hat{\jmath}\,'} = f_s\hat{\imath}\,' + N_y\hat{\jmath}\,'.$$



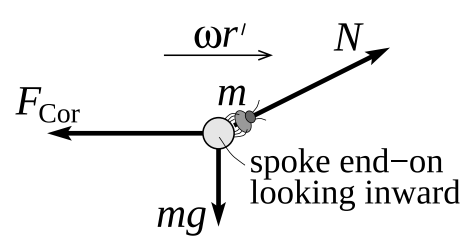
b) If static coefficient of friction is μ_s, how far can bug crawl before slipping?

Including $N_z = mg$,

$$N = \sqrt{N_z^2 + N_y^2} = \sqrt{m^2g^2 + 4m^2\omega^2 v_0^2}$$

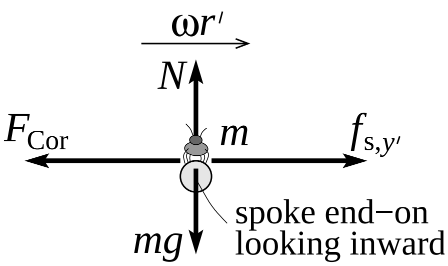
$$\Rightarrow \quad f_{s,\max} = \mu_s N = \mu_s m\sqrt{g^2 + 4\omega^2 v_0^2}$$

$$= F_{\text{cent}} = m\omega^2 r'_{\max}$$



$$\Rightarrow \quad \boxed{r'_{\max} = \frac{\mu_s\sqrt{g^2 + 4\omega^2 v_0^2}}{\omega^2}.} \qquad \square$$

This differs from solution in example 5.3.2 (p. 199): $r'_{\max} = \dfrac{\sqrt{\mu_s^2 g^2 - 4\omega^2 v_0^2}}{\omega^2}$.



As pointed out by Logan Francis ('14), if bug remains on top surface of spoke:

$$N = mg \quad \Rightarrow \quad f_{s,\max} = \mu_s N = \mu_s mg,$$

which balances $\vec{F}_{\text{Cor}}$ and $\vec{F}_{\text{cent}}$ at $r' = r'_{\max}$.

Thus, $\mu_s mg = \sqrt{4m^2\omega^2 v_0^2 + m^2\omega^4 r'^2_{\max}}$. Solving for $r'_{\max}$, we get:

$$\cancel{m^2}\omega^4 r'^2_{\max} = \mu_s^2 \cancel{m^2} g^2 - 4\cancel{m^2}\omega^2 v_0^2 \quad \Rightarrow \quad \boxed{r'_{\max} = \frac{\sqrt{\mu_s^2 g^2 - 4\omega^2 v_0^2}}{\omega^2},}$$

which does agree with example 5.3.2.

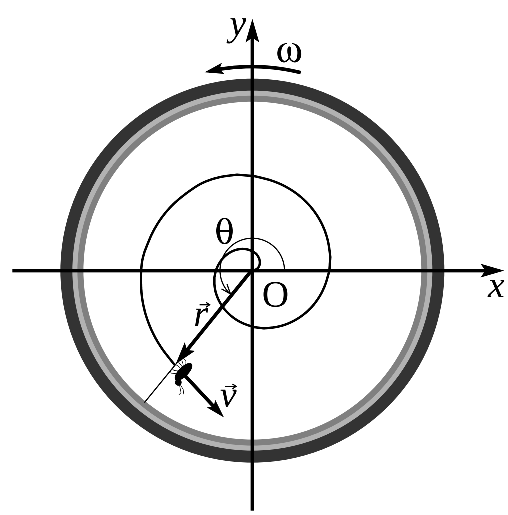
c) Repeat part b from an inertial reference frame, O.

From O, bug follows a spiral path. In polar coordinates (Eq. 1.3.1, 1.3.2),

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = v_0\hat{e}_r + r\omega\hat{e}_\theta,$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$
$$= -r\omega^2\hat{e}_r + 2v_0\omega\hat{e}_\theta,$$

where $\dot{r} = v_0$, $\dot{\theta} = \omega$, and $\ddot{r} = \ddot{\theta} = 0$.

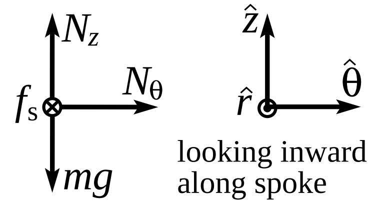


Thus, from FBD as viewed along end of spoke toward axle, Newton's 2nd law broken up into components is:

$$z/ \qquad N_z - mg \;=\; 0$$

$$r/ \quad -f_\mathrm{s} \;=\; ma_r \;=\; -mr\omega^2$$

$$\theta/ \quad N_\theta \;=\; ma_\theta \;=\; 2mv_0\omega$$

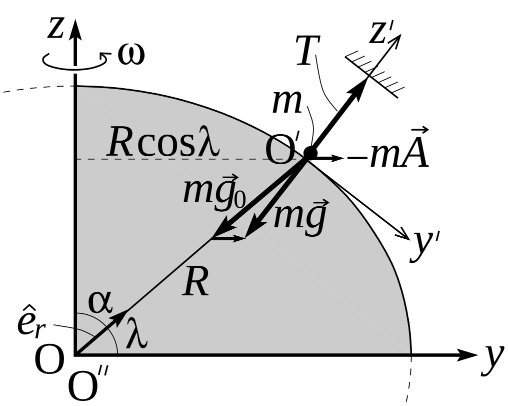


Thus, $f_{\mathrm{s,\,max}} \;=\; \mu_\mathrm{s} N \;=\; \mu_\mathrm{s}\sqrt{N_z^2 + N_\theta^2} \;=\; \mu_\mathrm{s} m\sqrt{g^2 + 4v_0^2\omega^2} \;=\; mr_\mathrm{max}\omega^2.$

$$\Rightarrow \quad \boxed{r_\mathrm{max} \;=\; \frac{\mu_\mathrm{s}\sqrt{g^2 + 4v_0^2\omega^2}}{\omega^2},} \quad \text{as above.} \qquad \square$$

4.4 Effects of Earth's rotation (FC §5.4)

Consider a "plumb-bob" of mass m hanging "vertically" downward at latitude λ.



Place O at centre of earth, O′ at m. Thus, $\vec{r}\,' = 0$, $\vec{v}\,' = 0$, and $\dot{\vec{\omega}} = 0$.

$$\Rightarrow \quad \vec{F}_\perp \;=\; \underbrace{\vec{F}_\mathrm{Cor}}_{\sim\vec{\omega}\times\vec{v}\,'} \;=\; \underbrace{\vec{F}_\mathrm{cent}}_{\sim\vec{\omega}\times\vec{r}\,'} \;=\; 0,$$

leaving (in Eq. 4.3.1) only real forces (T, $m\vec{g}_0$) and $\vec{F}_\mathrm{tr} = -m\vec{A}$.

Here, $\vec{g}_0 \;=\; -\dfrac{GM}{R^2}\hat{e}_r$ (points to centre of earth);

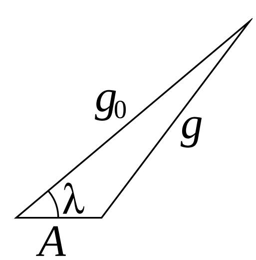
$\vec{A} \;=\; -\omega^2 R\cos\lambda\,\hat{\jmath}$ (*centripetal* acceleration points to rotation axis).

Define *effective acceleration of gravity*, $\vec{g} = \vec{g}_0 - \vec{A}$, to be actual acceleration of gravity, $\vec{g}_0$, plus *translational* (not *centrifugal*!) inertial acceleration.

$$\Rightarrow \quad \vec{g}\cdot\vec{g} \;=\; g^2 \;=\; g_0^2 - 2g_0 A\cos\lambda + A^2$$

$$\Rightarrow \quad g \;=\; \sqrt{g_0^2 - 2g_0\omega^2 R\cos^2\!\lambda + \omega^4 R^2\cos^2\!\lambda}$$

$$= \; g_0\sqrt{1 - (2-\beta)\beta\cos^2\!\lambda},$$



where $\beta \equiv \omega^2 R/g_0$.

In mks, $g_0 = \dfrac{GM}{R^2} = \dfrac{(6.674\times10^{-11})(5.974\times10^{24})}{(6.371\times10^{6})^2} = 9.823$, and thus[21],

$$\beta = \frac{(7.292\times10^{-5})^2\,(6.371\times10^{6})}{9.823} = 3.449\times10^{-3}.$$

$$\begin{aligned}
\text{Thus, at } \lambda &= 90^\circ \quad (\cos^2\lambda = 0), \quad g = g_0\times 1.0 \quad\;\; = 9.823 \quad \text{(poles)};\\
\lambda &= 45^\circ \quad (\cos^2\lambda = \tfrac{1}{2}), \quad g = g_0\times 0.9983 = 9.806 \quad \text{(Halifax)};\\
\lambda &= 0^\circ \quad (\cos^2\lambda = 1), \quad g = g_0\times 0.9966 = 9.789 \quad \text{(equator)}.
\end{aligned}$$

(150 pound person weighs $\frac{1}{2}$ pound less at equator than at a pole.)

Let $\vec{F}$ be all real forces except gravity. Then, on Earth's surface where $\dot{\vec{\omega}} = 0$, Eq. (4.3.1) becomes:

$$\begin{aligned}
\vec{F}' &= \vec{F} + \underbrace{m\vec{g}_0 - m\vec{A}}_{m\vec{g}} + \vec{F}_{\text{Cor}} + \vec{F}_{\text{cent}}\\
&= \vec{F} + m\vec{g} - 2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') = m\vec{a}'
\end{aligned}$$

$$\Rightarrow \quad \boxed{m\ddot{\vec{r}}' = \vec{F} + m\big(\vec{g} - 2\vec{\omega}\times\dot{\vec{r}}' - \vec{\omega}\times(\vec{\omega}\times\vec{r}')\big).} \qquad (4.4.1)$$

Aside: Why is correction to $\vec{g}$ (*i.e.*, $-m\vec{A}$) *translational* and not *centrifugal*?

- depends completely on where O′ is placed.
- placed at m where $\vec{r}' = 0$ and $\vec{v}' = 0$; O′ sees no centrifugal nor Coriolis force on m. Further, $\vec{a}' = 0$.
- O′ does accelerate relative to O: $\vec{A} = -\omega^2 R\cos\lambda\,\hat{\jmath}$ (*centripetal*); whence $\vec{F}_{\text{tr}} = -m\vec{A} = m\omega^2 R\cos\lambda\,\hat{\jmath}$; *looks like* (but isn't) a centrifugal force.

[21] Note that text value for ω (7.272×10^{-5}) is wrong in the third significant figure; evidently F&C use a solar rather than a sidereal day.

Consider now O$''$ rotating with earth whose origin coincides with O.

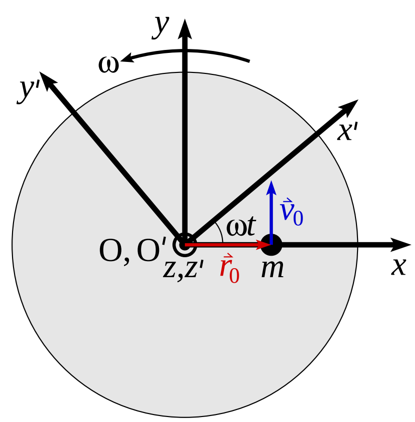
- O$''$ not accelerating relative to O $\Rightarrow$ $\vec{F}_{\rm tr} = 0$.
- $\vec{v}'' = 0$ ($\Rightarrow$ $F''_{\rm Cor} = 0$), but $\vec{r}'' = R \neq 0$ $\Rightarrow$

$$\vec{F}_{\rm cent} = -m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}'') = -m\omega^2 R\,\hat{k} \times (\hat{k} \times \hat{e}_r) = m\omega^2 R\,\hat{k} \times (\sin\alpha\,\hat{\imath})$$
$$= m\omega^2 R\cos\lambda\,\hat{\jmath}\,\Big(= -m\vec{A}\ \text{ worked out for O}'\text{ frame.}\Big)$$

Thus, $\left\{\begin{array}{l}\text{O}'\text{ interprets inertial force as translational}\\ \text{O}''\text{ interprets inertial force as centrifugal}\end{array}\right\}$ $\Rightarrow$ same result.

Tutorial 8.3

Problem 4: A merry-go-round with a frictionless ice surface rotates with a constant angular velocity, ω. Relative to an inertial (non-rotating) reference frame, O, whose origin is at the rotation axis, a hockey puck is placed at position $r_0\hat{\imath}$ and given an initial velocity $v_0\hat{\jmath}$.



a) In a reference frame O$'$ rotating with the merry-go-round and whose origin coincides with O, show that the equations of motion for the puck are:

$$\ddot{x}' = \omega^2 x' + 2\omega\dot{y}'; \qquad \ddot{y}' = \omega^2 y' - 2\omega\dot{x}', \quad (4.4.2)$$

two coupled, second order differential equations. Yikes!

b) Write down the equations of motion of the puck relative to O, and solve these to find $x(t)$ and $y(t)$.

c) By performing the appropriate coordinate transformation between O and O$'$, find $x'(t)$ and $y'(t)$, and show that these solve differential equations (4.4.2).

Lesson 21

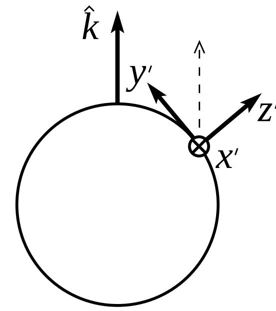
In this penultimate lesson, we re-examine projectile motion in two different accelerating frames of reference.

1. We examine how the Earth's rotation affects projectile motion (in the absence of air resistance), and find that in the limit of "slow" rotation ($r'\omega^2 \ll g$), projectiles are deflected measurably only by the Coriolis force.

2. We consider the science-fictional case of the *O'Neill Cylinder*, a huge ($R = 100\,\mathrm{km}$) hollow and closed cylinder in orbit about the Earth or sun rotating once on its axis every ten minutes inside which civilisations could live on the inner surface of the cylinder with an apparent g of 10. How can projectile motion be described in this reference frame?

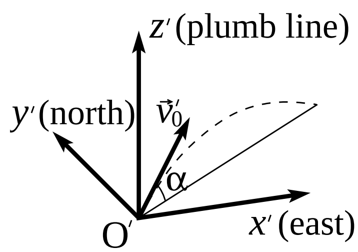
4.4.1 Projectile motion revisited

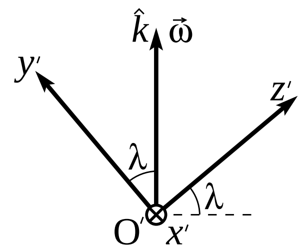
Choose coordinates as shown; set $\vec{F} = 0$ (no air resistance).



For ω small, ignore term $\vec{\omega} \times (\vec{\omega} \times \vec{r}') \propto \omega^2$
($r'\omega^2 \sim 5.3 \times 10^{-6}$ for $r \sim 1$ km $\ll g \sim 9.8$).

"Hunting and gathering" for Eq. (4.4.1):





$$\begin{aligned}\vec{g} &= -g\hat{k}'; \\ \vec{\omega} &= \omega(0, \cos\lambda, \sin\lambda); \\ \vec{r}' &= (x', y', z'); \\ \dot{\vec{r}}' &= (\dot{x}', \dot{y}', \dot{z}')\end{aligned}$$

$$\Rightarrow \quad \vec{\omega} \times \dot{\vec{r}}' = \omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda,\ \dot{x}'\sin\lambda,\ -\dot{x}'\cos\lambda). \qquad (4.4.3)$$

Thus, components of equation Eq. (4.4.1) are:

$$\left.\begin{aligned}\ddot{x}' &= 2\omega(\dot{y}'\sin\lambda - \dot{z}'\cos\lambda); \\ \ddot{y}' &= -2\omega\dot{x}'\sin\lambda; \\ \ddot{z}' &= -g + 2\omega\dot{x}'\cos\lambda.\end{aligned}\right\} \quad \text{(coupled)} \qquad (4.4.4)$$

Integrate last two of Eq. (4.4.4) once with respect to t to get:

$$\left.\begin{aligned}\dot{y}' &= -2\omega x'\sin\lambda + \dot{y}'_0; \\ \dot{z}' &= -gt + 2\omega x'\cos\lambda + \dot{z}'_0,\end{aligned}\right\} \qquad (4.4.5)$$

where $\vec{v}'_0 = (\dot{x}'_0, \dot{y}'_0, \dot{z}'_0)$ is initial velocity of projectile relative to O′.

Substitute Eq. (4.4.5) into first of Eq. (4.4.4) to get:

$$\ddot{x}' = 2\omega gt\cos\lambda - 2\omega(\dot{z}'_0\cos\lambda - \dot{y}'_0\sin\lambda),$$

dropping the two terms $\propto \omega^2$. Integrate to get:

$$v'_x = \dot{x}' = \dot{x}'_0 - 2\omega t(\dot{z}'_0 \cos\lambda - \dot{y}'_0 \sin\lambda) + \omega g t^2 \cos\lambda,$$

and again to get:

$$\boxed{x'(t) = x'_0 + \dot{x}'_0 t - \omega t^2(\dot{z}'_0 \cos\lambda - \dot{y}'_0 \sin\lambda) + \tfrac{1}{3}\omega g t^3 \cos\lambda,} \tag{4.4.6}$$

where $\vec{r}'_0 = (x'_0, y'_0, z'_0)$ is initial position of projectile relative to O′.

Finally, substitute Eq. (4.4.6) into each of Eq. (4.4.5) and integrate to get:

$$\boxed{\begin{aligned} y'(t) &= y'_0 + \dot{y}'_0 t - \omega \dot{x}'_0 t^2 \sin\lambda; \\ z'(t) &= z'_0 + \dot{z}'_0 t - \tfrac{1}{2} g t^2 + \omega \dot{x}'_0 t^2 \cos\lambda, \end{aligned}} \tag{4.4.7}$$

ignoring terms $\propto \omega^2$.

- Leading terms in Eq. (4.4.6), (4.4.7) are from projectile motion in an inertial frame
- terms $\propto \omega$ are "correction terms" needed in a (slowly) rotating frame

Exercise: Fill in the details in derivation of Eq. (4.4.6) and (4.4.7).

Example 4.6. Drop m from rest at height h. Where does it land?

Solution: $\vec{v}'_0 = (0,0,0)$, $\vec{r}'_0 = (0,0,h)$, and Eq. (4.4.6)–(4.4.7) become:

Hidden segment

Hidden segment

Example 4.7. Fire a rifle horizontally to the east with muzzle speed v_0. Where does projectile land?

Solution: $\vec{v}_0' = (v_0, 0, 0)$ and take $\vec{r}_0' = (0, 0, h)$. Then, Eq. (4.4.6), (4.4.7) $\Rightarrow$

$$x'(t) = v_0 t + \tfrac{1}{3}\omega g t^3 \cos\lambda \quad \Rightarrow \text{ moves further than expected;}$$
$$y'(t) = -\omega v_0 t^2 \sin\lambda \quad \Rightarrow \text{ drifts S (N) for } \lambda > 0\ (< 0);$$
$$z'(t) = h - \tfrac{1}{2}(g - 2\omega v_0 \cos\lambda)t^2 \quad \Rightarrow \text{ effective } g \text{ less; takes longer to fall.}$$

Time to fall, t_f, found by setting $z' = 0$: $t_f = \sqrt{\dfrac{2h}{g - 2\omega v_0 \cos\lambda}}$.

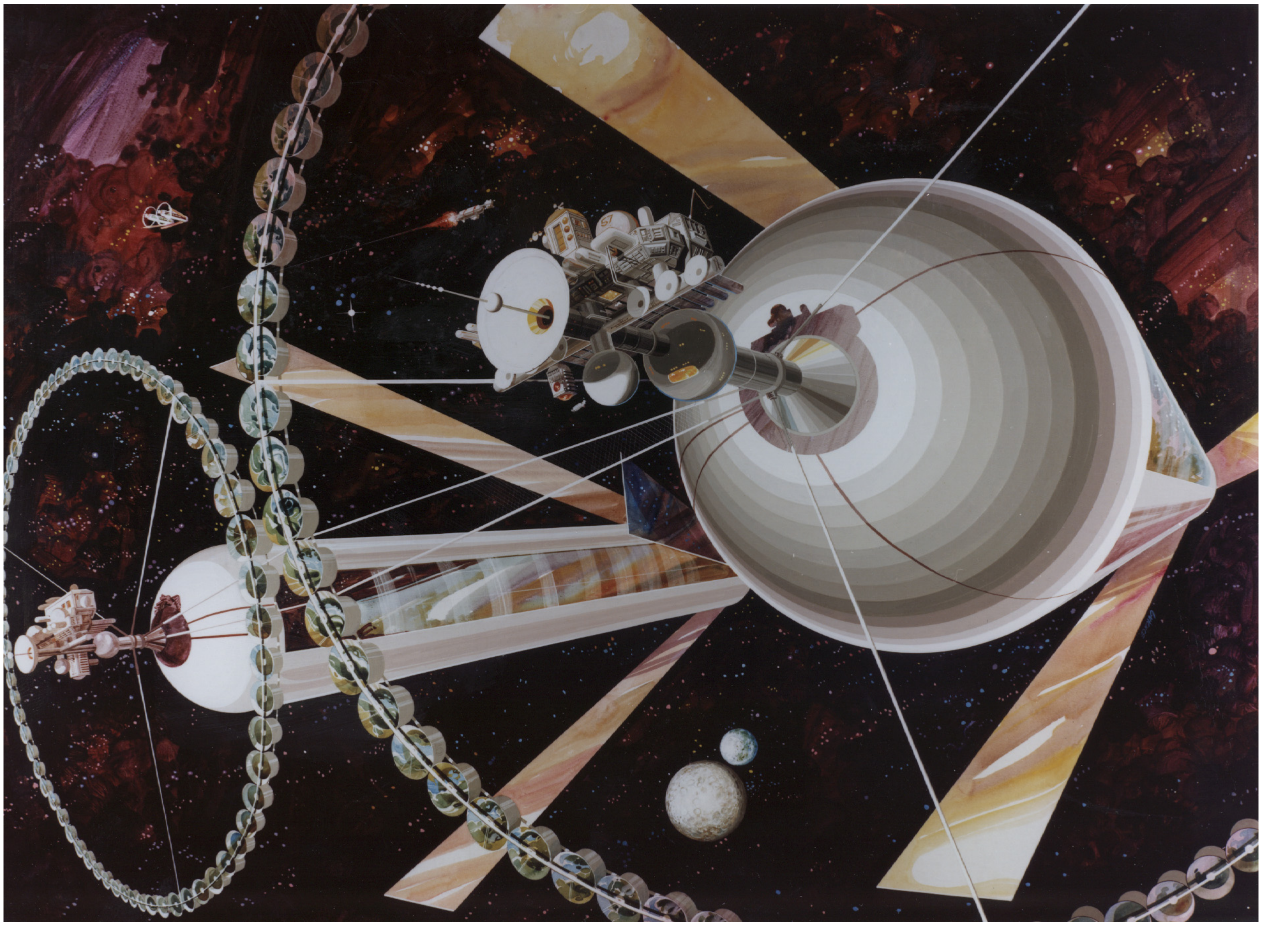
For $h = 2\,\text{m}$, $v_0 = 10^3\,\text{m/s}$, $\omega = 7.29 \times 10^{-5}\,\text{rad/s}$, $g = 9.81\,\text{m/s}^2$ at $\lambda = 45°$, we get $t_f = 0.642\,\text{s}$ (0.638 s ignoring correction). The range, R, is:

$$R = x'(t_f) \approx v_0 t_f = 642\,\text{m},$$

ignoring the t^3 term (7 parts in 10^8). The deflection, Δ, is:

$$\Delta = y'(t_f) = -\omega v_0 t_f^2 \sin\lambda = -0.021\,\text{m (2.1 cm south)}. \qquad \square$$

4.5 The O'Neill cylinder (FC §5.5)



(public domain image from NASA/Rick Guidice)

The *O'Neill cylinder* is a hypothetical giant rotating cylinder in deep space capable of sustaining life—even entire civilisations—on its inner surface.

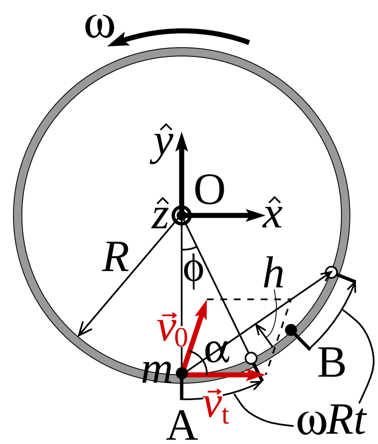
- proposed by Gerard O'Neill in *High Frontier: Human Colonies in Space*
- featured in Arthur C. Clarke's *Rendez-vous with Rama*

$R \sim 100\,\mathrm{km}$ and $\omega \sim 10^{-2}\,\mathrm{rad\,s^{-1}}$ ($T \sim 10\,\mathrm{min}$) gives an "effective gravity" $g = \omega^2 R \sim 10\,\mathrm{m\,s^{-2}}$ for inhabitants living on inside surface.

Alas, it seems two factions in our space colony don't get along very well, and are lobbing catapults at each other. . .

Let cylinder radius be R and angular speed about z-axis be ω. Thus, tangential speed of cylindrical surface is $v_{\mathrm{t}} = \omega R$.

A launches projectile mass m, velocity $\vec{v}_0$ (relative to A) at B. Relative to inertial frame, O,



- No real forces ⇒ m has constant velocity:

  $$\vec{v} \;=\; \vec{v}_0 + \vec{v}_{\mathrm{t}} \;=\; (v_0 \cos\alpha + \omega R)\hat{x} + v_0 \sin\alpha\,\hat{y};$$

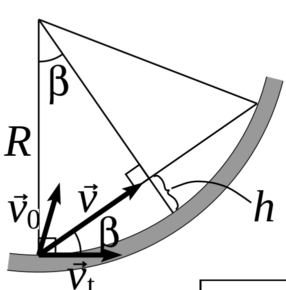
- In time t, A and B rotate by angle $\phi = \omega t$;

- In same time, m travels distance vt to hit B.

Example 4.8. Find maximum "height", h, m gets from inner surface.

Solution: Let β be angle between $\vec{v}$ and $\vec{v}_t$ when m is launched. Then,

$$\cos\beta \;=\; \frac{\vec{v}_{\mathrm{t}} \cdot \vec{v}}{|\vec{v}_{\mathrm{t}}||\vec{v}|} \;=\; \frac{\cancel{\omega R}(v_0 \cos\alpha + \omega R)}{\cancel{\omega R}\sqrt{(v_0 \cos\alpha + \omega R)^2 + (v_0 \sin\alpha)^2}}$$



$$= \frac{v_0 \cos\alpha + \omega R}{\sqrt{v_0^2 + 2v_0 \omega R \cos\alpha + \omega^2 R^2}} \;=\; \frac{\cos\alpha + \xi}{\sqrt{1 + 2\xi\cos\alpha + \xi^2}},$$

where $\xi = \omega R / v_0$. Then, from the figure:

$$\boxed{h \;=\; R(1 - \cos\beta) \;=\; R\left(1 - \frac{\cos\alpha + \xi}{\sqrt{1 + 2\xi\cos\alpha + \xi^2}}\right).}$$

Example 4.9. Derive equations of motion of projectile as viewed from rotating reference frame A.

Solution Step 1: "Hunting and gathering".

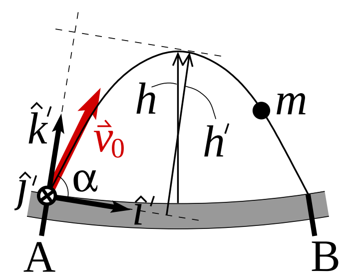
angular vel. of A relative to O: $\vec{\omega} = \omega\hat{z} = -\omega\hat{\jmath}'$

position of m relative to A: $\vec{r}' = x'\hat{\imath}' + z'\hat{k}'$;

velocity of m relative to A: $\vec{v}' = \dot{x}'\hat{\imath}' + \dot{z}'\hat{k}'$;

acceleration of m relative to A: $\vec{a}' = \ddot{x}'\hat{\imath}' + \ddot{z}'\hat{k}'$;

acceleration of A relative to O: $\vec{A}_0 = \omega^2 R\hat{k}'$.



Step 2: Do the cross products.

Coriolis: $-2m\vec{\omega}\times\vec{v}' = 2m\omega\hat{\jmath}'\times(\dot{x}'\hat{\imath}' + \dot{z}'\hat{k}') = 2m\omega(-\dot{x}'\hat{k}' + \dot{z}'\hat{\imath}')$;

centrifugal: $-m\vec{\omega}\times(\vec{\omega}\times\vec{r}') = -m\omega^2\hat{\jmath}'\times\left(\hat{\jmath}'\times(x'\hat{\imath}' + z'\hat{k}')\right)$

$$= -m\omega^2\hat{\jmath}'\times(-x'\hat{k}' + z'\hat{\imath}') = m\omega^2(x'\hat{\imath}' + z'\hat{k}') = m\omega^2\vec{r}'.$$

Step 3: Assemble the forces (Eq. 4.3.1:)

$$\vec{F}' = m\vec{a}' = \cancelto{0}{\vec{F}} - m\cancelto{0}{\dot{\vec{\omega}}}\times\vec{r}' - 2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') - m\vec{A}_0$$

$$\Rightarrow \quad \ddot{x}'\hat{\imath}' + \ddot{z}'\hat{k}' = 2\omega(-\dot{x}'\hat{k}' + \dot{z}'\hat{\imath}') + \omega^2(x'\hat{\imath}' + z'\hat{k}') - \omega^2 R\hat{k}'$$

$$\Rightarrow \quad \ddot{x}' = 2\omega\dot{z}' + \omega^2 x' \qquad \text{and} \qquad \ddot{z}' = -2\omega\dot{x}' + \omega^2(z' - R).$$

In the limit $R \gg z', x'$ and $\omega R \gg \dot{x}', \dot{z}'$, these reduce to:

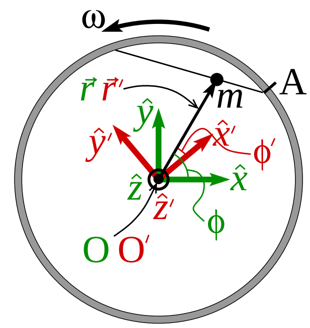
$$\ddot{x}' \approx 0 \qquad \text{and} \qquad \ddot{z}' \approx -\omega^2 R \sim -g,$$

and we have same ballistics problem as we do near earth's surface.

Tutorial 8.4

Problem 5 (FC 5.12) A particle moves in a horizontal plane near the surface of the Earth. Show that the magnitude of the horizontal component of the Coriolis force is independent of the direction of motion.

Problem 6 Consider again the O'Neill cylinder discussed in class and the same inhabitants launching projectiles at each other.



This time, consider the problem from the rotating reference frame O′ which, as shown in the figure, is cospatial with inertial reference frame O. If O′ rotates with the cylinder such that $\hat{x}'$ always points to A (fixed to the inside surface), show that the equations of motion for a projectile launched from A (with no external real forces acting on it) are given by:

$$\ddot{r}' = r'(\omega + \dot{\phi}')^2; \qquad \ddot{\phi}' = -\frac{2\dot{r}'}{r'}(\omega + \dot{\phi}'); \qquad \ddot{z}' = 0.$$

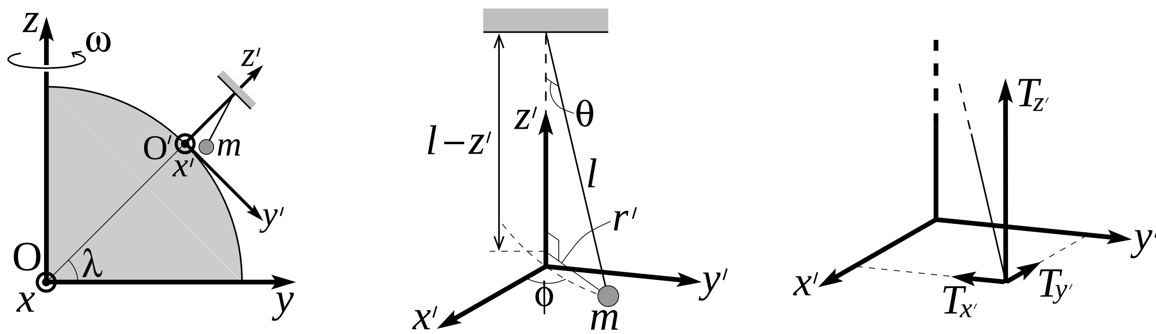
Lesson 22

This lesson wraps up the course by examining the historically important *Foucault pendulum* as our final example of motion in an accelerating frame of reference. The equations of motion lead to a rather tricky pair of ODEs whose solution shows that the plane of swing of the bob *precesses* once every 24-hours divided by the sine of the latitude, and that this effect is entirely due to the Coriolis force.

When first constructed in Paris in 1851, there was still sufficient lingering doubt among the masses that the Earth actually rotated on its axis, and this was taken—even by the public—as incontrovertible proof that it did.

4.6 The Foucault (spherical) pendulum (FC §5.6)

Consider a *Foucault pendulum* (Léon Foucault, 1819–1868), whose plane of rotation rotates freely about vertical axis passing through its support.



From Eq. (4.4.1), we have (ignoring terms $\propto \omega^2$ and thus centrifugal):

$$m\ddot{\vec{r}}' = \vec{T} + m\vec{g} - 2m\vec{\omega} \times \dot{\vec{r}}', \tag{4.6.1}$$

The (x', y', z') components of $\vec{T}$ in terms of θ and ϕ are:

$$\vec{T} = T(-\sin\theta\cos\phi,\ -\sin\theta\sin\phi,\ \cos\theta),$$

where, from the diagram:

$$\cos\theta = \frac{l - z'}{l}; \quad \sin\theta = \frac{r'}{l}; \quad \cos\phi = \frac{x'}{r'}; \quad \sin\phi = \frac{y'}{r'}.$$

Thus,

$$\vec{T} = T\underbrace{\left(-\frac{x'}{l}, -\frac{y'}{l}, \frac{l - z'}{l}\right)}_{\text{directional cosines}} = -\frac{x'}{l}T\hat{\imath}' - \frac{y'}{l}T\hat{\jmath}' + \frac{l - z'}{l}T\hat{k}'.$$

Next, $\vec{g} = -g\hat{k}'$ and, from Eq. (4.4.3; projectiles), we have:

$$\vec{\omega} \times \dot{\vec{r}}' = \omega(\dot{z}'\cos\lambda - \dot{y}'\sin\lambda)\hat{\imath}' + \omega\dot{x}'\sin\lambda\,\hat{\jmath}' - \dot{x}'\cos\lambda\,\hat{k}'.$$

Thus, components of Eq. (4.6.1) are:

$$\ddot{x}' = -\frac{x'}{l}\frac{T}{m} + 2\dot{y}'\omega\sin\lambda - 2\dot{z}'\omega\cos\lambda; \tag{4.6.2}$$

$$\ddot{y}' = -\frac{y'}{l}\frac{T}{m} - 2\dot{x}'\omega\sin\lambda; \tag{4.6.3}$$

$$\ddot{z}' = \frac{l - z'}{l}\frac{T}{m} - g + 2\dot{x}'\omega\cos\lambda. \tag{4.6.4}$$

For small angles ($\theta \ll 1$), vertical speed $\dot{z}' \approx 0 \;\Rightarrow\; \ddot{z}' \approx 0,\; (l - z')/l \approx 1$.

Further, for $\dot{x}' \sim 0.1\,\mathrm{m/s}$, $2\dot{x}'\omega\cos\lambda \sim 10^{-5} \ll g \sim 10$.

Thus, Eq. (4.6.4) $\Rightarrow$ $T/m \approx g$, and Eq. (4.6.2), (4.6.3) become:

$$\ddot{x}' \approx -x'\frac{g}{l} + 2\dot{y}'\omega_z; \tag{4.6.5}$$

$$\ddot{y}' \approx -y'\frac{g}{l} - 2\dot{x}'\omega_z, \tag{4.6.6}$$

where $\omega_z \equiv \omega\sin\lambda$, component of $\vec{\omega}$ in $\hat{k}'$ direction.

Terms $\propto g/l$ make Eq. (4.6.5), (4.6.6) more difficult to solve than projectile equations (Eq. 4.4.4).

"Standard trick": add $\sqrt{-1} = i$ times Eq. (4.6.6) to Eq. (4.6.5) to get:

$$\ddot{x}' + i\ddot{y}' + (x' + iy')\frac{g}{l} = 2\omega_z(\dot{y}' - i\dot{x}') = -2i\omega_z(\dot{x}' + i\dot{y}')$$

$$\Rightarrow \quad \ddot{q}' + 2i\omega_z\dot{q}' + \Omega^2 q' = 0, \tag{4.6.7}$$

where $q' \equiv x' + iy'$ and $\Omega^2 \equiv g/l$.

This is just the damped harmonic oscillator equation with an imaginary damping constant, $2i\omega_z$, whose solution is:

$$q'(t) = e^{-i\omega_z t}\left(Ae^{i\sqrt{\omega_z^2+\Omega^2}\,t} + Be^{-i\sqrt{\omega_z^2+\Omega^2}\,t}\right), \tag{4.6.8}$$

where A and B are set from initial conditions (Eq. 2.4.2).

Exercise: Verify by direct substitution that Eq. (4.6.8) solves Eq. (4.6.7).

Consequences of *imaginary* damping constant:

- renders oscillatory what was the exponential decay factor, $e^{-\gamma t} \to e^{-i\omega_z t}$;
- renders argument under radical negative definite, $\gamma^2 - \omega_0^2 \to -\omega_z^2 - \Omega^2$, and thus terms $\propto A$ and B are *always* oscillatory.

Now, $\omega_z^2 < (7.29 \times 10^{-5})^2 \sim 5 \times 10^{-9} \ll \Omega^2 \sim 1$ for $l \sim 10\,\text{m}$. Thus,

$$\text{Eq. (4.6.8)} \quad \Rightarrow \quad \boxed{q'(t) \;=\; e^{-i\omega_z t}\left(Ae^{i\Omega t} + Be^{-i\Omega t}\right).} \tag{4.6.9}$$

To interpret this solution, consider first an inertial frame in which $\omega_z = 0$:

$$\begin{aligned} \text{Eq. (4.6.9)} \quad \Rightarrow \quad q(t) \;&=\; Ae^{i\Omega t} + Be^{-i\Omega t} \\ &=\; A(\cos\Omega t + i\sin\Omega t) + B(\cos\Omega t - i\sin\Omega t) \\ &=\; (A+B)\cos\Omega t + i(A-B)\sin\Omega t \;=\; x(t) + iy(t) \end{aligned} \tag{4.6.10}$$

$$\Rightarrow \quad x(t) \;=\; (A+B)\cos\Omega t; \quad y(t) \;=\; (A-B)\sin\Omega t,$$

equating real and imaginary parts. Thus,

$$\frac{x^2}{(A+B)^2} + \frac{y^2}{(A-B)^2} \;=\; 1,$$

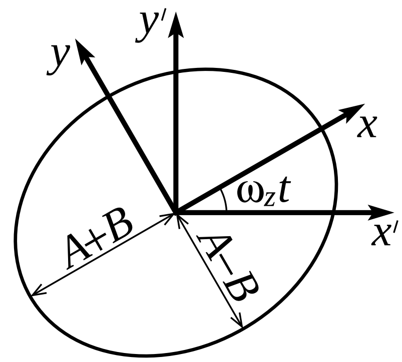
an ellipse with semi-major (semi-minor) axis $A+B$ $(A-B)$.

Without dissipation, m traces over the same ellipse forever.

Back to the rotating frame, substitute Eq. (4.6.10) into (4.6.9):

$$\begin{aligned} q'(t) \;&=\; e^{-i\omega_z t}q(t) \;=\; (\cos\omega_z t - i\sin\omega_z t)\big(x(t) + iy(t)\big) \\ &=\; \cos\omega_z t\, x(t) + \sin\omega_z t\, y(t) + i\big(-\sin\omega_z t\, x(t) + \cos\omega_z t\, y(t)\big) \\ &=\; x'(t) + iy'(t) \quad \Rightarrow \quad \begin{cases} x'(t) \;=\; \cos\omega_z t\, x(t) + \sin\omega_z t\, y(t) \\ y'(t) \;=\; -\sin\omega_z t\, x(t) + \cos\omega_z t\, y(t) \end{cases} \end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \cos\omega_z t & \sin\omega_z t \\ -\sin\omega_z t & \cos\omega_z t \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.}$$



Path traced out by m in O$'$ is same as in O but rotated by angle $\omega_z t$.

On earth, spherical pendulum traces out an ellipse (straight line for $A = B$) that precesses at frequency $\omega_z = \omega \sin\lambda$.

Period of precession: $T = \dfrac{2\pi}{\omega\sin\lambda} = \dfrac{24 \text{ hr}}{\sin\lambda} = 33.9$ hr at $\lambda = 45°$.

YouTube videos for Foucault pendulum:

1. Animation of Foucault's pendulum;
2. Foucault's pendulum in the Pantheon, Paris.
3. Kurtis Baute's "Watch the World Spin!".

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