PHYS 2302 SECOND PRACTISE MIDTERM

Instructions

There are three problems on this exam, each worth 20 points. Do as much of this exam as you can; your mark will be the sum of all your points taken out of 45.

This is a closed-book exam; notebooks, textbooks, *etc.*, are not permitted. You may use a simple scientific calculator, but all other electronic devices should be put away.

Formulæ are provided on the last three pages of the exam. Including the formula sheets, there are six pages to this exam. You may tear off the formula sheets from the exam for ready reference if you like.

Solutions are to be written in the exam booklet provided. This exam, including the formula sheets, must be turned in with your solutions.

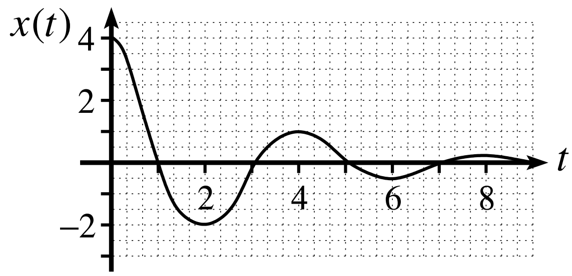
You have 75 minutes.

Problem 1. Damped harmonic oscillator: The plot shows the position of an under-damped harmonic oscillator as a function of time. The units on the axes are arbitrary but, if it helps, you can think of the units on the ordinate as cm and those on the abscissa as s.

As given on the formula sheets, the equation of motion for such an oscillator is,

$$x(t) = x_0 e^{-\gamma t} \frac{\omega_0}{\omega_\mathrm{d}} \cos(\omega_\mathrm{d} t - \theta_0), \qquad (1)$$

where all quantities are defined on the formula sheets.



a) (11 points; 3 for γ, 2 for each of the others) By reading off whatever data you may need from the plot, find numerical values good to three significant figures for each of the five constants in equation (1).

Caution: Read off only values that can be read accurately (such as the period, T). Values that cannot be read accurately from the plot (such as the phase, θ_0) should be calculated from appropriate expressions using values that can be read accurately.

b) (9 points) Using your quantities in part a, find values good to three significant figures for each of:

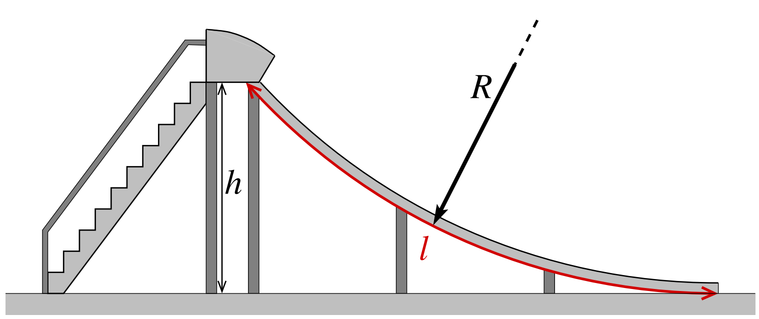
i) the resonant frequency, ω_r;

ii) the quality factor, Q_d;

iii) the resonant amplitude, $A_{\max}$, for $F_0/m = 10.0$ in the arbitrary units of the plot.

Caution: In order to ensure accuracy to three significant figures, your input values from part a will have to be good to 4 or even 5 significant figures to avoid accumulating round-off error in the third significant figure reported.

Problem 2. A playground slide: The figure depicts a playground slide in the form of an arc of a circle with a height $h = 4.00\,\mathrm{m}$, radius of curvature $R = 12.0\,\mathrm{m}$, and with the ground tangent to the circular arc. A 25.0 kg child starts from rest at the top of the slide and has a speed of $6.20\,\mathrm{m\,s^{-1}}$ at the bottom.



a) Show that the arc-length of the slide is given by: $l = R\cos^{-1}\dfrac{R-h}{R}$.

b) Use the work-kinetic theorem to find the average friction force acting on the child by the slide, where the average value of a quantity, q, over a path P of length l is,

$$\langle q\rangle_{\mathrm{P}} \equiv \frac{1}{l}\int_{\mathrm{P}} q\,dl. \tag{1}$$

c) What is the maximum normal force exerted by the slide on the child?

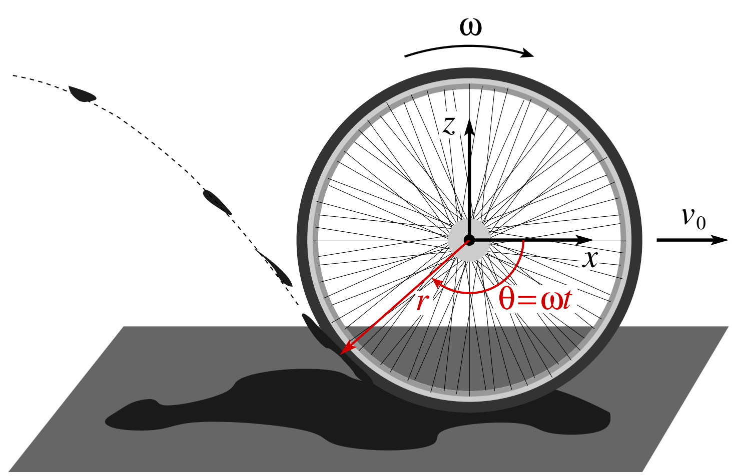
Give numerical values for each answer accurate to three significant figures.

Problem 3. Mud flung from a tire:

a) (16 points) As shown in the figure, blobs of mud are thrown from the rim of a rolling bicycle wheel of radius r. If the forward speed of the wheel axle is v_0 and constant, show that the greatest height *above the ground* that the mud can attain is given by,

$$h_{\max} = r + \frac{v_0^2}{2g} + \frac{r^2 g}{2v_0^2}.$$

Hint: This is a projectile problem, where the height attained by a blob of mud (the projectile) needs to be maximised with respect to the launching angle, θ.



b) (4 points) If $r = 0.36$ m and $v_0 = 6\ \mathrm{m\,s^{-1}}$, at what value of θ does the mud attaining the greatest height leave the wheel?

FORMULA SHEETS

1. **Period of oscillation:** If the angular frequency of oscillation is ω (rad s^{-1}), the period of oscillation in seconds is:

$$T = \frac{2\pi}{\omega}.$$

2. **Damped harmonic oscillator:** The differential equation of motion for a damped harmonic oscillator is,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \tag{1}$$

where: γ = damping coefficient;

ω_0 = angular frequency of the undamped oscillator.

For initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$, the solution to equation (1) is:

$$x(t) = \begin{cases} x_0 e^{-\gamma t}\left(\cosh qt + \dfrac{\gamma}{q}\sinh qt\right), & \gamma > \omega_0 \text{ (overdamped)}; \\ x_0 e^{-\gamma t}(1 + \gamma t), & \gamma = \omega_0 \text{ (critically damped)}; \\ x_0 e^{-\gamma t}\,\dfrac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \theta_0), & \gamma < \omega_0 \text{ (underdamped)}; \\ x_0 \cos\omega_0 t, & \gamma = 0 \text{ (undamped)}, \end{cases}$$

where: x_0 = initial displacement of oscillator;

$q = \sqrt{\gamma^2 - \omega_0^2}$;

$\omega_\mathrm{d} = \sqrt{\omega_0^2 - \gamma^2}$ = angular frequency of damped oscillator;

$\theta_0 = \sin^{-1}\dfrac{\gamma}{\omega_0}$ = phase lag of damped oscillator.

3. **Quality factor of a damped harmonic oscillator**, Q_d, is a measure of how slowly the total energy of an oscillator is lost to damping. For underdamped systems,

$$Q_\mathrm{d} \approx \frac{\omega_\mathrm{d}}{2\gamma},$$

where: ω_d = angular frequency of the damped oscillator;

$\gamma = \sqrt{\omega_0^2 - \omega_\mathrm{d}^2}$, the damping coefficient;

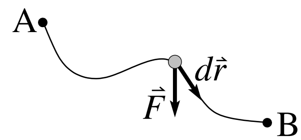
ω_0 = angular frequency of the undamped oscillator.

4. **Resonance:** Maximum amplitude of a driven, damped oscillator is: $A_{\text{max}} = \dfrac{F_0}{2m\gamma\omega_{\text{d}}}$, when driven at the *resonant frequency*,

$$\omega_{\text{r}} = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\omega_{\text{d}}^2 - \gamma^2},$$

where: F_0 = amplitude of the driving force;
m = mass of the oscillator;
γ = damping coefficient;
$\omega_{\text{d}} = \sqrt{\omega_0^2 - \gamma^2}$ = natural angular frequency of damped oscillator;
ω_0 = natural angular frequency of undamped oscillator.

5. **Work done by a force along a path**:



$$W = \int_{\text{A}}^{\text{B}} \vec{F} \cdot d\vec{r}.$$

If $\vec{F}$ is conservative, then W depends only on the end points A and B, independent of the path chosen between them.

6. **Work-kinetic theorem**: Work done by all external forces along a path between points A and B is equal to the difference in kinetic energy at points B and A:

$$\sum W = \Delta K \quad \Rightarrow \quad \sum_i \int_{\text{A}}^{\text{B}} \vec{F}_{\text{ext},i} \cdot d\vec{r} = K_{\text{B}} - K_{\text{A}}.$$

7. **Centripetal acceleration:** For an object moving at speed v in a circular path of radius r, its centripetal acceleration directed toward the centre of curvature is,

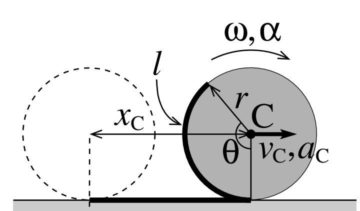
$$a_{\text{cp}} = \frac{v^2}{r} = \omega^2 r,$$

where $\omega = v/r$ is the angular speed.

8. **Rolling motion (no-slip condition)**: Let C correspond to the axle of the wheel. Then, for the wheel to roll without slipping,

$$x_{\text{C}} = l = r\theta; \qquad v_{\text{C}} = r\omega; \qquad a_{\text{C}} = r\alpha,$$

where: x_{C}, v_{C}, and a_{C} are the distance traveled, translational speed, and acceleration of the axle respectively; r is the radius of the wheel; $\omega = \dot{\theta}$, $\alpha = \ddot{\theta}$ are the angular speed, acceleration of the wheel about C.



9. **Projectile motion**. In the absence of air resistance, the position components of a projectile are given by:

$$\begin{aligned} x(t) &= x_0 + (v_0 \cos\alpha)t; & \text{(horizontal component)} \\ z(t) &= z_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2, & \text{(vertical component)} \end{aligned}$$

where:

$$\begin{aligned} (x_0, 0, z_0) &= \text{initial position of the projectile;} \\ v_0 &= \text{initial speed of the projectile;} \\ \alpha &= \text{angle between } \vec{v}_0 \text{ and the horizontal in the } x\text{-}z \text{ plane.} \end{aligned}$$

By setting $x_0 = 0$ and eliminating t between the x- and z-components, one gets the trajectory equation:

$$z(x) = z_0 + x\tan\alpha - \frac{g}{2v_0^2\cos^2\alpha}x^2.$$

10. **Extremising a function**: To find the extrema of a function $f(x)$, set $\dfrac{df(x)}{dx} = 0$.

The values of x that solve this equation are x_{ext}, the locations of the extrema. The function extrema are then given by $f(x_{\text{ext}})$.

The nature of each extremum is determined by the second derivative:

$$\left.\frac{d^2f(x)}{dx^2}\right|_{x=x_{\text{ext}}} \begin{cases} < 0, & \text{maximum;} \\ = 0, & \text{inflection point;} \\ > 0, & \text{minimum.} \end{cases}$$