Practise Final Exam Solutions

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1. Damped, driven harmonic oscillator: A damped harmonic oscillator is set into motion and it is found that after an integer n of oscillations, its amplitude is half the initial amplitude. The oscillator is then driven by a driving force $F = F_0 \cos \omega t$ for varying ω and, from a plot of the square of its amplitude A^2 *vs.* the driving frequency ω, both half-peak frequencies, ω_+ and ω_- (see formula sheets), are measured. Thus, the quantity,

$$\Delta\omega^2 = \omega_+^2 - \omega_-^2,$$

is known. In terms of just n, $\Delta\omega^2$, and/or various numerical constants:

a) find the quality factor, Q_d;

b) find the damping coefficient, γ;

c) find the natural oscillation period of the damped harmonic oscillator, ω_d; and

d) show that the resonant frequency, ω_r, is given by:

$$\omega_\mathrm{r}^2 \;=\; \frac{\Delta\omega^2}{4}\left(\frac{2\pi n}{\ln 2} - \frac{\ln 2}{2\pi n}\right).$$

e) If the damped oscillator is a mass m on a spring with spring constant k, what is k in terms of n, $\Delta\omega^2$, and m?

Recall for a Hooke's spring, the undamped frequency of oscillation is given by $\omega_0^2 = \dfrac{k}{m}$.

Solution: a) From the formula sheets, the equation of motion for an undriven, underdamped harmonic oscillator is,

$$x(t) \;=\; x_0 e^{-\gamma t}\frac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \phi),$$

where $\cos\phi = \dfrac{\omega_\mathrm{d}}{\omega_0}$. Thus, at $t = 0$, $x(0) = x_0$ and at time $t = nT_\mathrm{d}$ later,

$$x(nT_\mathrm{d}) \;=\; x_0 e^{-\gamma n T_\mathrm{d}}\frac{\omega_0}{\omega_\mathrm{d}}\cos(n\omega_\mathrm{d} T_\mathrm{d} - \phi) \;=\; \frac{x_0}{2}. \tag{1}$$

But,

$$\omega_\mathrm{d} T_\mathrm{d} \;=\; 2\pi \quad\Rightarrow\quad \cos(n\omega_\mathrm{d} T_\mathrm{d} - \phi) \;=\; \cos(n2\pi - \phi) \;=\; \cos\phi \;=\; \frac{\omega_\mathrm{d}}{\omega_0},$$

and equation (1) becomes:

$$x_0 e^{-\gamma n T_\mathrm{d}}\frac{\cancel{\omega_0}}{\cancel{\omega_\mathrm{d}}}\frac{\cancel{\omega_\mathrm{d}}}{\cancel{\omega_0}} \;=\; \frac{x_0}{2} \quad\Rightarrow\quad e^{-\gamma n T_\mathrm{d}} \;=\; \frac{1}{2} \quad\Rightarrow\quad \gamma n T_\mathrm{d} \;=\; \ln 2$$

$$\Rightarrow \quad \gamma T_\mathrm{d} = \frac{2\pi\gamma}{\omega_\mathrm{d}} = \frac{\pi}{Q_\mathrm{d}} = \frac{\ln 2}{n} \quad \Rightarrow \quad \boxed{Q_\mathrm{d} = \frac{n\pi}{\ln 2}.} \tag{2}$$

b) From the formula sheets,

$$\Delta\omega^2 = \omega_+^2 - \omega_-^2 = 4\gamma\omega_\mathrm{d} \quad \Rightarrow \quad \omega_\mathrm{d} = \frac{\Delta\omega^2}{4\gamma}. \tag{3}$$

Now, from equation (2), we have,

$$\frac{2\pi\gamma}{\omega_\mathrm{d}} = \frac{\ln 2}{n} \quad \Rightarrow \quad \omega_\mathrm{d} = \frac{2\pi\gamma n}{\ln 2}.$$

Comparing this with equation (3), we get:

$$\frac{2\pi\gamma n}{\ln 2} = \frac{\Delta\omega^2}{4\gamma} \quad \Rightarrow \quad \boxed{\gamma = \frac{1}{2}\sqrt{\frac{\Delta\omega^2 \ln 2}{2\pi n}}.} \tag{4}$$

c) Substituting equation (4) into equation (3), we get,

$$\omega_\mathrm{d} = \frac{\Delta\omega^2}{4} 2\sqrt{\frac{2\pi n}{\Delta\omega^2 \ln 2}} \quad \Rightarrow \quad \boxed{\omega_\mathrm{d} = \frac{1}{2}\sqrt{\frac{2\pi n \Delta\omega^2}{\ln 2}}.}$$

d) From the formula sheets,

$$\omega_\mathrm{r}^2 = \omega_\mathrm{d}^2 - \gamma^2 = \frac{1}{4}\frac{2\pi n\Delta\omega^2}{\ln 2} - \frac{1}{4}\frac{\Delta\omega^2 \ln 2}{2\pi n} \quad \Rightarrow \quad \boxed{\omega_\mathrm{r}^2 = \frac{\Delta\omega^2}{4}\left(\frac{2\pi n}{\ln 2} - \frac{\ln 2}{2\pi n}\right),}$$

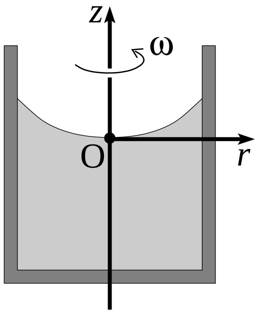
as desired.

e) As for k, we first need an expression for ω_0^2. From the formula sheets,

$$\omega_0^2 = \omega_\mathrm{d}^2 + \gamma^2 = \frac{\Delta\omega^2}{4}\left(\frac{2\pi n}{\ln 2} + \frac{\ln 2}{2\pi n}\right) = \frac{k}{m}$$

$$\Rightarrow \quad \boxed{k = \frac{m\Delta\omega^2}{4}\left(\frac{2\pi n}{\ln 2} + \frac{\ln 2}{2\pi n}\right).}$$

Problem 2. Mach bucket: A bucket of water spins at an angular speed ω about its symmetry axis. As shown in the figure, an inertial coordinate system, O, is defined such that z is aligned with the symmetry axis and r is the radial distance from the axis. You are to find the function, $z(r)$, that describes the spinning surface of the water in two ways.



a) (12 points) Assess all real and inertial forces acting on a droplet of water of mass m, say, at rest on the rotating surface using a rotating coordinate system, O′, whose origin and z'-axis coincide with the origin and z-axis of O, and whose r' axis rotates with the water so that m remains at rest in the r'-z' plane.

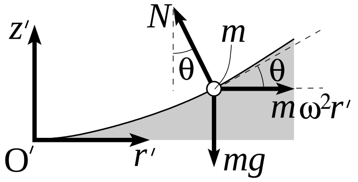
   Hint: If θ is the tangent angle to the water surface at m, $\tan\theta = dz'/dr' = dz/dr$.

b) (8 points) Assess only the real forces acting on m from the inertial coordinate system, O, and once again find $z(r)$.

Obviously, the two methods should give identical results.

Solution: a) To assess the real and inertial forces acting on m from the coordinate system O′ as described in the problem and shown in the figure to the right, start with the force equation from the formula sheet:

$$\begin{aligned}\vec{F}' &= \vec{F} - m\dot{\vec{\omega}}\times\vec{r}\,' - 2m\vec{\omega}\times\vec{v}\,' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,') - m\vec{A} \\ &= m\vec{a}\,'.\end{aligned} \tag{5}$$



Step 1. "Hunting and gathering". angular velocity of O′ relative to O: $\vec{\omega} = \omega\hat{e}_{z'}$;
position of m relative to O′: $\vec{r}\,' = r'\hat{e}_{r'} + z'\hat{e}_{z'}$.

All other kinematical quantities of interest are zero, namely $\dot{\vec{\omega}}$, $\vec{v}\,'$ (m at rest in $\hat{e}_{r'}$-$\hat{e}_{z'}$ plane), $\vec{A}$ (O and O′ coincide), and $\vec{a}\,'$. As indicated in the FBD, the real forces acting on m are:

$$\vec{F} = -N\sin\theta\,\hat{e}_{r'} + (N\cos\theta - mg)\hat{e}_{z'}. \tag{6}$$

Step 2: Do the cross products. The only non-zero cross-product is the centrifugal force:

$$-m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,') = -m\omega^2\,\hat{e}_{z'}\times\big(\hat{e}_{z'}\times(r'\hat{e}_{r'} + z'\hat{e}_{z'})\big) = m\omega^2 r'\hat{e}_{r'}, \tag{7}$$

as indicated on the FBD.

Step 3: Assemble the forces. Substitute equations (6) and (7) into equation (5) to get:

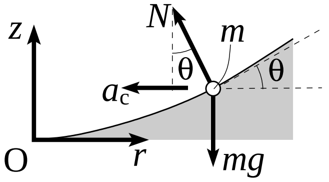
$$-N\sin\theta\,\hat{e}_{r'} + (N\cos\theta - mg)\hat{e}_{z'} + m\omega^2 r'\hat{e}_{r'} = 0$$

$$\Rightarrow \quad \begin{Bmatrix} r'/ & N\sin\theta = m\omega^2 r' \\ z'/ & N\cos\theta = mg \end{Bmatrix} \quad \Rightarrow \quad \tan\theta = \frac{\omega^2 r'}{g} = \frac{dz'}{dr'},$$

using the hint. Because of azimuthal symmetry, we can rewrite this in terms of the unprimed coordinates, and thus:

$$\frac{dz}{dr} = \frac{\omega^2 r}{g} \quad \Rightarrow \quad \boxed{z(r) = \frac{\omega^2 r^2}{2g},} \tag{8}$$

given $z = 0$ at $r = 0$, as implied by the figure.



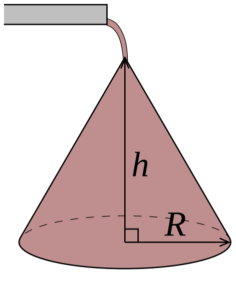
b) In the inertial frame of reference, O, the FBD does not include the centrifugal force. Instead, the droplet m has a centripetal acceleration, a_c, pointing toward the z axis, and Newton's second law is:

$$\sum \vec{F} = m\vec{a} \quad \Rightarrow \quad -N\sin\theta\,\hat{e}_r + (N\cos\theta - mg)\hat{e}_z = -m\omega^2 r\hat{e}_r$$

$$\Rightarrow \quad \left\{ \begin{aligned} r/ \quad N\sin\theta &= m\omega^2 r \\ z/ \quad N\cos\theta &= mg \end{aligned} \right\} \quad \Rightarrow \quad \tan\theta = \frac{\omega^2 r}{g} = \frac{dz}{dr},$$

giving us exactly the conditions that lead to equation (8).

Problem 3. A cone-shaped sandpile: A conveyor belt piles sand into a "right cone" of height h and base radius R, as depicted in the figure.



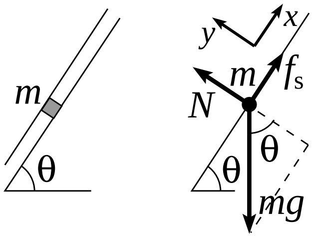
If the coefficient of static friction between each layer of sand along the slope and the sand beneath it (along which the sand might slip) is μ_s, what is the maximum volume of sand in the cone for a given R if none of the sand slips? Your final answer should be in terms of μ_s and R.

Hint: The volume of a right cone is $\frac{\pi}{3}R^2 h$.

Solution: Consider the top-most layer of sand, and a portion of that layer of mass m as shown in the figure. Then, from the FBD for m, we have from Newton's second law:

$$x/ \quad f_s - mg\sin\theta = 0;$$

$$y/ \quad N - mg\cos\theta = 0,$$



for no acceleration. Now, for a given R, to maximise the volume we must maximise h. Thus, the sand must be on the verge of slipping, in which case $f_s = \mu_s N$ and we have:

$$\mu_s N = mg\sin\theta \quad \Rightarrow \quad \mu_s \cancel{mg}\cos\theta = \cancel{mg}\sin\theta \quad \Rightarrow \quad \tan\theta = \mu_s.$$

On the other hand,

$$\tan\theta = \frac{h}{R} \quad\Rightarrow\quad h = R\tan\theta = R\mu_{\rm s}.$$

Then from the hint, the volume of the cone is given by:

$$V = \frac{\pi}{3}R^2 h \quad\Rightarrow\quad \boxed{V_{\rm max} = \frac{\pi}{3}R^2\mu_{\rm s}R = \frac{\pi}{3}\mu_{\rm s}R^3.}$$

Problem 4. Air drag on a bullet: A gun is fired vertically upward, and the bullet experiences an air drag given by $f_{\rm d} = bv^2$, where v is the speed of the bullet, b is a constant, and where $f_{\rm d}$ is directed opposite to the direction of motion. Let z be the vertical coordinate.

a) (7 points) Ignoring the effects of earth's rotation, show that the function $v_{\rm up}(z)$ of the bullet on its way up is given by:

$$v_{\rm up}^2(z) = Ae^{-2kz} - \frac{g}{k},$$

where $k = b/m$ and A is an arbitrary constant.

Hint: From an FBD, show that Newton's second law can be written as the separable first order ODE, $\frac{1}{2}d\phi/dz = -k\phi - g$, where $\phi = v^2$. Then, if you've forgotten how to solve a separable ODE, see the reminder on the formula sheets.

b) (5 points) Show that the function $v_{\rm down}(z)$ of the bullet on its way down is given by:

$$v_{\rm down}^2(z) = Be^{2kz} + \frac{g}{k},$$

where B is another arbitrary constant.

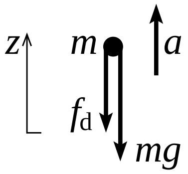
c) (5 points) If the muzzle speed of the bullet is v_0, find A and B.

Hint: To find A, set $v_{\rm up}(0) = v_0$. To find B, note that $v_{\rm up}(h) = v_{\rm down}(h) = 0$.

d) (3 points) What is the speed of the bullet when it returns to the height of the muzzle on its way down? (*Hint*: It's not v_0!)

Solution: a) As the bullet rises, we see from the FBD that:

$$-f_{\rm d} - mg = -bv^2 - mg = ma = \frac{m}{2}\frac{dv^2}{dz},$$



from the formula sheet. Setting $\phi = v^2$ and $k = b/m$, this gives:

$$\frac{1}{2}\frac{d\phi}{dz} = -k\phi - g, \tag{9}$$

a separable first order ODE. Thus,

$$\frac{d\phi}{k\phi+g} = -2dz \quad \Rightarrow \quad \int \frac{k d\phi}{k\phi+g} = -2k\int dz \quad \Rightarrow \quad \ln(k\phi+g) = -2kz+c,$$

where c is a constant of integration. Thus,

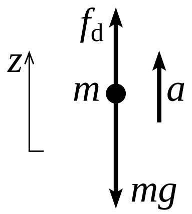
$$k\phi+g = e^{-2kz+c} = e^c e^{-2kz} \quad \Rightarrow \quad \phi(z) = \frac{e^c}{k}e^{-2kz} - \frac{g}{k}.$$

Defining $A = e^c/k$, the general solution to equation (9) is:

$$\Rightarrow \quad \boxed{\phi(z) = v_{\text{up}}^2(z) = Ae^{-2kz} - \frac{g}{k},} \tag{10}$$

as desired.

b) Similar to part a, as the bullet falls we see from the FBD that:



$$f_{\text{d}} - mg = bv^2 - mg = ma = \frac{m}{2}\frac{dv^2}{dz}$$

$$\Rightarrow \quad \frac{1}{2}\frac{d\phi}{dz} = k\phi - g. \tag{11}$$

where once again, $\phi = v^2$ and $k = b/m$.

As the only difference between equations (9) and (11) is the sign on k, we can immediately write down the solution to equation (11) by changing the sign of k in equation (10):

$$\boxed{\phi(z) = v_{\text{down}}^2(z) = Be^{2kz} + \frac{g}{k},} \tag{12}$$

where B is a constant of integration, as desired.

c) If $v(z=0) = v_0$, equation (10) $\Rightarrow$

$$v_0^2 = A - \frac{g}{k} \quad \Rightarrow \quad \boxed{A = v_0^2 + \frac{g}{k}.}$$

As for B, at the top of the trajectory where $z = h$, both equations (10) and (12) should give $v = 0$. Thus, from equation (10) we have:

$$0 = \left(v_0^2 + \frac{g}{k}\right)e^{-2kh} - \frac{g}{k} \quad \Rightarrow \quad e^{2kh} = \frac{v_0^2 + g/k}{g/k} = \frac{kv_0^2+g}{g}. \tag{13}$$

Then, from equation (12), we have:

$$0 = Be^{2kh} + \frac{g}{k} = B\frac{kv_0^2+g}{g} + \frac{g}{k} \quad \Rightarrow \quad \boxed{B = -\frac{g}{k}\frac{g}{kv_0^2+g},} \tag{14}$$

using equation (13).

d) To find v of the bullet when it falls back down to the level of the muzzle, set $z = 0$ in equation (12) and use equation (14) for B:

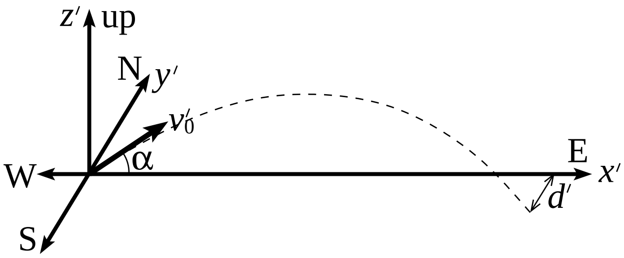
$$\boxed{v^2_{\text{down}}(z=0) \;=\; -\frac{g}{k}\,\frac{g}{kv_0^2+g}+\frac{g}{k} \;=\; \frac{g}{k}\,\frac{kv_0^2}{kv_0^2+g},}$$

which, for $kv_0^2 \gg g$ as is typical, is just g/k, the terminal velocity squared (*e.g.*, set $d\phi/dz = 0$ in eq. 11). Thus, v on the way down has very little to do with the muzzle speed, v_0.

Problem 5. Inertial deflection of a bullet: A bullet is fired from a rifle with a muzzle speed v_0' due east at latitude λ and an elevation angle α (angle between the rifle's long axis and the horizontal).

a) (15 points) Ignoring air resistance and how gravity varies with height, show that the bullet hits the earth with a lateral deflection, d', given by:

$$d' \;=\; -\sqrt{\frac{2R_0^3\tan\alpha}{g}}\,\omega\sin\lambda,$$



dropping all terms proportional to ω^2 or higher power. Here, the negative sign means the drift is southward, $R_0 = (v_0'^2\sin 2\alpha)/g$ is the rifle's range not accounting for the earth's rotation, and ω is the rotation speed of the earth.

b) (5 points) For $v_0' = 500\,\text{m}\,\text{s}^{-1}$, $\alpha = 30°$, and $\lambda = 45°$, find a numerical value for d'/R_0, the deflection as a fraction of its approximate range. Take the earth's rotation speed to be $7.292\times10^{-5}\,\text{rad}\,\text{s}^{-1}$ and $g = 9.81\,\text{m}\,\text{s}^{-2}$.

Solution: a) We're not concerned with the range of the bullet, just its lateral drift. Thus, consider only the north-south and vertical components of a projectile near the surface of the earth,

$$\left.\begin{aligned} y'(t)-y_0' &= \dot y_0' t - \omega t^2 \dot x_0'\sin\lambda; \\ z'(t)-z_0' &= \dot z_0' t - \tfrac12 g t^2 + \omega t^2 \dot x_0'\cos\lambda. \end{aligned}\right\} \tag{15}$$

Using initial conditions $(x_0', y_0', z_0') = (0,0,0)$, and $(\dot x_0', \dot y_0', \dot z_0') = (v_0'\cos\alpha, 0, v_0'\sin\alpha)$, equations (15) become:

$$y'(t) \;=\; -\omega t^2 v_0'\cos\alpha\sin\lambda; \tag{16}$$

$$z'(t) \;=\; v_0' t\sin\alpha - \tfrac12 g t^2 + \omega t^2 v_0'\cos\alpha\cos\lambda. \tag{17}$$

When the bullet hits the ground, $z'(t) = 0$, and equation (17) $\Rightarrow$ $t = 0$, or

$$t\left(\tfrac{1}{2}g - \omega v_0' \cos\alpha\cos\lambda\right) = v_0'\sin\alpha \quad\Rightarrow\quad t = \frac{v_0'\sin\alpha}{\frac{1}{2}g - \omega v_0'\cos\alpha\sin\lambda} \simeq \frac{2v_0'\sin\alpha}{g}, \qquad (18)$$

where we drop the correction term proportional to ω in the denominator because $y'(t)$ is already proportional to ω, and we are dropping all terms proportional to ω^2 or higher.

Substituting equation (18) into equation (16), we get:

$$\begin{aligned} y'(t) &= d' = -\omega\left(\frac{2v_0'\sin\alpha}{g}\right)^2 v_0'\cos\alpha\sin\lambda = -\frac{4v_0'^3\sin^2\!\alpha\cos\alpha}{g^2}\,\omega\sin\lambda \\ &= -\sqrt{\frac{2\sin\alpha}{g\cos\alpha}}\;\underbrace{\frac{v_0'^3\big(\overbrace{2\sin\alpha\cos\alpha}^{\sin 2\alpha}\big)^{3/2}}{g^{3/2}}}_{R_0^{3/2}}\,\omega\sin\lambda = \boxed{-\sqrt{\frac{2R_0^3\tan\alpha}{g}}\,\omega\sin\lambda,} \end{aligned}$$

as desired.

b) Using the numbers given,

$$\begin{aligned} \frac{d'}{R_0} &= -\sqrt{\frac{2R_0\tan\alpha}{g}}\,\omega\sin\lambda = \sqrt{\frac{2v_0'^2 2\sin\alpha\cos\alpha\tan\alpha}{g^2}}\,\omega\sin\lambda = -\frac{2v_0'\sin\alpha}{g}\,\omega\sin\lambda \\ &\sim -\frac{2(500)\sin 30^\circ}{9.81}\,(7.292\times 10^{-5})\sin 45^\circ \sim \underline{\underline{2.628\times 10^{-3}}}, \end{aligned}$$

or about 0.26%. For a range $R_0 \sim 22\,\text{km}$, this amounts to a southward deflection of $\sim$58 m.
