Extra Problems

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1 (FC 1.9) Prove the so-called "back-cab rule": $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$.

Since this is a *vector* identity, it must be true for *any* coordinate system. Designing a proof around expanding $\vec{A}$, $\vec{B}$, and $\vec{C}$ in their Cartesian coordinates and then proving LHS = RHS will only prove the assertion for Cartesian coordinates, and is not a general proof!

I'll give full points for a general proof that does not rely on expanding $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Cartesian coordinates. *Hint*: The vector $\vec{A}\times(\vec{B}\times\vec{C})$, let's call it $\vec{D}$, is perpendicular to $\vec{B}\times\vec{C}$ and thus lies in the plane spanned by $\vec{B}$ and $\vec{C}$. Thus, we should be able to write $\vec{D}$ as a linear combination of $\vec{B}$ and $\vec{C}$...

If you need to bunt, I'll give half points to a solution restricted to Cartesian (or any other specific) coordinates.

Problem 2 (FC 1.19) A bee's path is a spiral given in polar coordinates by:

$$r = be^{kt}; \qquad \theta = ct,$$

where b, c, and k are constant. Show that the angle between $\vec{v}$ and $\vec{a}$ is constant.

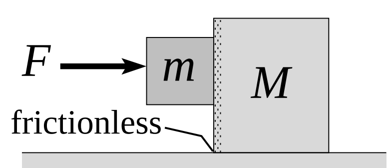
Problem 3 (FC 1.23, 1.24)

a) Prove that: $\dfrac{d}{dt}\big(\vec{r}\cdot(\vec{v}\times\vec{a})\big) = \vec{r}\cdot(\vec{v}\times\dot{\vec{a}})$.

b) Prove that $\vec{v}\cdot\vec{a} = v\dot{v}$, which means that if $|\vec{v}| = v =$ constant, then $\vec{a}\perp\vec{v}$. (*Hint:* try differentiating both sides of the identity $\vec{v}\cdot\vec{v} = v^2$, and note that $\dot{v}$ is *not* the same as $|\vec{a}|$! *Really?*, you might ask, *why not?*)

Problem 4 As shown in the figure, a block of mass m is pressed against a separate block of mass M by a horizontal force, F.

If the coefficient of static friction between the two blocks is μ_s and the surface on which M is resting is frictionless, find the minimum force, F, required to keep m from slipping along M. What is the acceleration of the two blocks when this minimum force is applied?

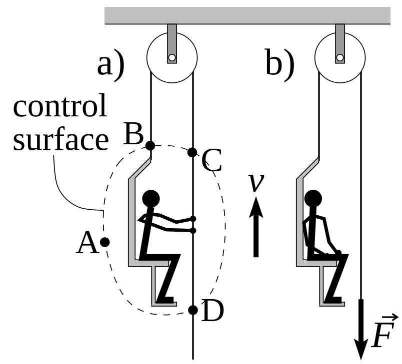


Problem 5 A "bosun's chair" is a chair or harness with a rope attached to the back which is then threaded through an overhead pulley. A person wishing to hoist themself upward

pulls on the dangling end of the rope as shown in the figure. Let the combined mass of the chair and person be m, and suppose the pulley and cord are "ideal" (massless, frictionless). Among other things, this means the tension everywhere along the rope is the same.

a) What force must the person exert on the rope to rise at a constant speed?

b) What force, F, must someone from below exert on the rope to raise m at a constant speed?



Hint: On problems involving pulleys and cords, a useful construct is a *control surface* that encloses all that the dot m is to represent on the free body diagram (FBD). Thus, as shown in the figure for part a, the control surface includes every bit of m, including the person's hands grasping the rope. (Don't cut off the hands!!)

Starting at point A, for example, move clockwise around the control surface and note on the FBD each and every *external force* exerted on m encountered until you return to A. Thus, a rope is encountered at B. Does it exert an external force? That is, is it under tension (*e.g.*, imagine "plucking" it)? If so, there is a tension force acting on m, whose direction needs to be determined before adding it to the FBD. What about C? What about D? And don't forget about gravity which acts through the control surface.

Problem 6 For each of the following first order ODEs, find $y(x)$ using separation of variables:

a) $(x^2+1)\dfrac{dy}{dx} - xy = 0;$

b) $e^{x+y}\dfrac{dy}{dx} + x = 0;$

c) $x\dfrac{dy}{dx} + x + y = 1.$

Hint: Turns out that as written, the ODE in part c is not separable. But did you notice that $xy' + y$ is a *perfect derivative*? That is, $xy' + y = (xy)'$? That would suggest a substitution of, say, $z = xy\ldots$

Problem 7 (FC 2.2) Find $v(x)$ for a mass, m, starting from rest at $x = 0$, subject to the forces:

a) $F_x = F_0 + cx;$

b) $F_x = F_0 e^{-cx};$

c) $F_x = F_0 \cos(cx)$.

Problem 8 A skier of mass m points her skis directly down the fall-line of a hill with a slope θ to the horizontal, and accelerates downward. At some point later, crouched down with her ski poles tucked under her arms, she attains a terminal velocity, v_t.

a) If the coefficient of friction between her skis and the snow is μ_k, use the work-kinetic theorem to find the power generated by the air drag, P_D, at terminal velocity.

b) If $\theta = 20°$, $\mu_\mathrm{k} = 0.15$, $mv_\mathrm{t} = 500\,\mathrm{kg\,m\,s^{-1}}$ (terminal momentum), and $g = 9.81\,\mathrm{m\,s^{-2}}$, find numerical values (to three significant figures) for the power generated by each of gravity, friction, and air drag in both Watts and horsepower ($1\,\mathrm{hp} = 745.7\,\mathrm{W}$). What does the sign of the power indicate?

Hint: *Power* is the rate at which work is done. Thus, if motion is along the x-axis and F_x is the x-component of a constant force, $\vec{F}$, the power generated by $\vec{F}$ is:

$$P_F = \frac{dW_F}{dt} = \frac{F_x dx}{dt} = F_x v, \tag{1}$$

where v is the speed of m in the x-direction. You will therefore wish to consider the *differential form* of the work-kinetic theorem, namely,

$$\sum dW = dK, \tag{2}$$

where $dW_F = F_x dx$ is the differential work done by the x-component of the force, F_x, over a differential distance, dx, and dK is the differential change in kinetic energy of m over dx.

Problem 9 (FC 2.6, 2.11)

a) (4 points) A mass, m, moves with velocity $v(x) = \alpha/x$ along a horizontal, frictionless surface where x is the distance from the origin and $\alpha > 0$ is a constant. Find $F(x)$ acting on the mass.

b) (6 points) Find the distance travelled by a block of mass m with an initial velocity v_0 sliding along an oil film that provides a viscous resistance force $F(v) = -cv^{3/2}$, where c is a constant.

Problem 10 (FC 2.18) The force acting on a particle of mass m is given by $F = kvx$, where $k > 0$ is a constant. If m passes through the origin ($x = 0$) with speed v_0 at $t = 0$, find $x(t)$, the position of m as a function of time.

Problem 11 Consider the following linear, second-order, homogeneous ODE:

$$y''(x) + 2y'(x) = 0. \tag{1}$$

a) Find "by inspection" two linearly independent solutions to equation (1).

b) From your two linearly independent solutions, write down the general solution.

c) Show that when the boundary conditions $y(0) = 1$ and $y'(0) = -1$ are applied to your general solution in part b, you get:

$$y(x) = e^{-x} \cosh x,$$

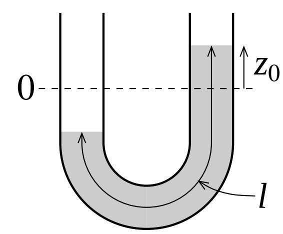
where $\cosh x \equiv \frac{1}{2}(e^x + e^{-x})$ is the *hyperbolic cosine function*, which we'll meet in an upcoming class.

Problem 12 (FC 3.1 & 3.3)

a) A guitar string vibrates harmonically at a frequency of 512 Hz (one octave above middle C). If the amplitude of oscillation at the centre-point of the string is 2.00 mm, what are the maximum speed and acceleration at that point?

b) A particle undergoes simple harmonic motion with a frequency of 10.0 Hz. Find the displacement at any time, $x(t)$, for the initial conditions ($t = 0$) $x(0) = 0.250\,\mathrm{m}$ and $\dot{x}(0) = 0.100\,\mathrm{m\,s^{-1}}$.

Problem 13 A U-tube with a uniform cross-section, A, is partially filled with water. When held vertically, the water level in the left side is depressed by a distance z_0 then released, causing the water to oscillate back and forth in the tube.



a) (7 points) If the mechanical energy of the system is conserved, show that the system oscillates as a simple harmonic oscillator with a period of oscillation given by,

$$T = \pi\sqrt{\frac{2l}{g}},$$

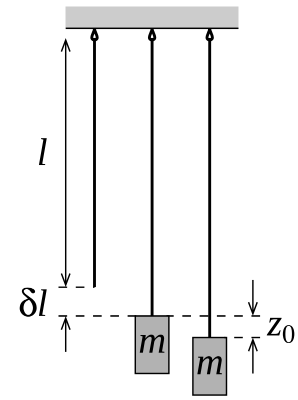
where l is the "column length" of the water, as shown in the inset.

b) (3 points) All of the water in the tube oscillates and thus accelerates, and the "m" in the equation of motion for a SHO,

$$m\ddot{z} + kz = 0,$$

is the mass of all the water in the tube. If the density of water is ρ, find the effective spring constant, k.

Problem 14 In this problem, you will learn about the elasticity of some materials, and an important example of simple harmonic motion not discussed in class. One might argue this falls under the realm of materials science or even engineering, but any good physicist should at least be aware of what is known as *Young's modulus*.



Consider a wire of length l, cross sectional area A, and negligible mass hanging vertically from a fixed anchor. If a mass m is hung from the free end, the wire stretches by an amount δl, as shown in the inset. Typically, $\delta l \ll l$ (*e.g.*, a thin metal wire will break before δl gets too big), although for a rubber band, δl could be comparable to and even greater than l.

Define the *strain*, Σ (the Greek capital 'S'), as the *distortion* of the wire caused by the weight of the hanging mass, m:

$$\Sigma \equiv \frac{\delta l}{l}, \tag{1}$$

a unitless quantity. Next, define the *stress* on the wire, S, as the *applied force per unit area* on the wire:

$$S \equiv \frac{F}{A}, \tag{2}$$

where $F = mg$ in this case. Thus, S has the units of *pressure*.

So far, everything has been definitions. Here's the only bit of physics: Experimentally, for *elastic* materials (which includes steel, by the way), *the stress is proportional to the strain*:

$$S \propto \Sigma \quad \Rightarrow \quad S = -Y\Sigma, \tag{3}$$

where the proportionality constant, $Y > 0$, is *Young's modulus* (units $\mathrm{N\,m^{-2}}$), a property of the material making up the wire (like its density or electrical conductivity). The negative sign means the stress and strain act in opposite directions. In the present example, the distortion is downward while the restoring force (the tension in the wire) acts upward.

Note that equation (3) is an approximation for how elastic materials actually behave, with the approximation better for smaller $\delta l/l$. Young's moduli for several common materials are given in the table below in units of GPa (1 gigaPascal = $10^9\,\mathrm{N\,m^{-2}}$).

rubber band	0.01	glass	70	copper	117
aluminum	69	brass	110	steel	200

a) If m is pulled down an additional distance $z_0 \ll l$ then released, show that in the absence of any dissipative (frictional) forces, m moves up and down as a simple harmonic oscillator with a period given by:

$$T = 2\pi\sqrt{\frac{ml}{YA}}. \tag{4}$$

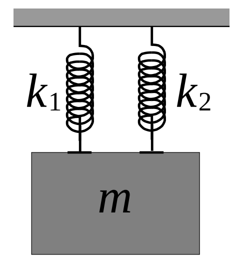
You may assume $\delta l \ll l$ and thus $l + \delta l \approx l$.

Hint: Review what we did in class for vertical oscillators (§2.2 in the class notes).

b) Find a numerical value for the period of oscillation for $m = 1.00\,\text{kg}$ hanging on a length $l = 1.00\,\text{m}$ of 12-gauge copper wire (2.05 mm in diameter). Why do you suppose this would be difficult to demonstrate in class?

c) By considering the equilibrium state of m hanging on the wire, show that the period given in equation (4) is the same as that of a simple pendulum of length δl.

Problem 15 (FC 3.7) Consider two springs of the same length with spring constants k_1 and k_2.



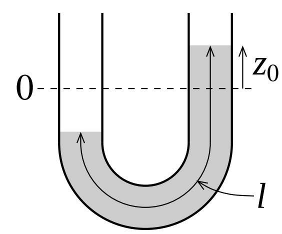
a) (4 points) The two springs are hung vertically *in parallel* and a mass, m is attached to both, as shown in the figure. Find the frequency of oscillations.

b) (6 points) Find the frequency of vertical oscillations if the springs are attached to m *in series* (*i.e.*, k_1 attached to the ceiling, k_2 attached to the bottom of k_1, and then m attached to the bottom of k_2).

Hint: Review the discussion on vertical springs in §2.2.

Hint for part b: You may wish to consider an FBD for both m *and* the massless point where the springs join.

Problem 16 Consider the U-tube problem of the previous assignment.



A U-tube with cross-sectional area A is partially filled with water of column length l. When held vertically, the water is set into oscillation with initial amplitude z_0 where it is noted that after 10 complete oscillations, the amplitude of oscillation has fallen by a factor of 2.

From equation (2.4.1) in the class notes, the equation of motion for a damped harmonic oscillator is: $\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = 0$. Using the result of the previous assignment, namely,

$$T_0 = \frac{2\pi}{\omega_0} = \pi\sqrt{\frac{2l}{g}}, \tag{1}$$

find the effective "damping coefficient", γ, for the oscillating water.

Problem 17 Starting from the general solution for the damped harmonic oscillator derived in class [equation (2.4.2) in the class notes], namely,

$$x(t) = e^{-\gamma t}\left(Ae^{qt} + Be^{-qt}\right), \tag{1}$$

where, as defined in class, $\gamma = b/2m$ is the damping coefficient, $q = \sqrt{\gamma^2 - \omega_0^2}$, and $\omega_0 = \sqrt{k/m}$ is the oscillator frequency when $b = 0$, use the initial conditions,

$$x(0) = 0 \qquad \text{and} \qquad \dot{x}(0) = v_0,$$

to show that,

$$x(t) = v_0 \begin{cases} e^{-\gamma t}\,\dfrac{\sinh qt}{q}, & \gamma > \omega_0 \text{ (overdamped);} \\ te^{-\omega_0 t}, & \gamma = \omega_0 \text{ (critically damped);} \\ e^{-\gamma t}\,\dfrac{\sin \omega_{\mathrm{d}} t}{\omega_{\mathrm{d}}}, & \gamma < \omega_0 \text{ (underdamped);} \\ \dfrac{\sin \omega_0 t}{\omega_0}, & \gamma = 0 \text{ (undamped).} \end{cases}$$

These initial conditions are tantamount to giving the oscillator a sharp blow (impulse) from the equilibrium point of the spring at $t = 0$, resulting in an initial velocity of v_0.

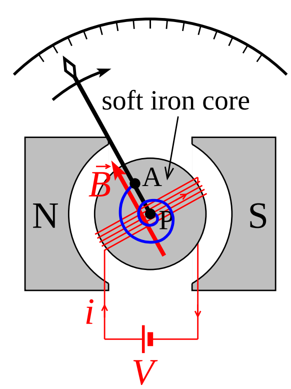
Problem 18 (FC 3.12) The frequency of a damped harmonic oscillator is $f_{\mathrm{d}} = 100\,\mathrm{Hz}$ and the ratio of the amplitude of two successive minima is $\frac{1}{2}$.

a) (7 points) What is the undamped frequency, f_0, of the oscillator?

b) (3 points) What is the quality factor, Q_{d}?

Hint: You may use the result of problem 3.9, namely that the ratio of successive minima of a damped oscillator is $e^{-\gamma T_{\mathrm{d}}}$, where $T_{\mathrm{d}} = 2\pi/\omega_{\mathrm{d}}$ is the period of oscillation.

Problem 19 A galvanometer is an analogue instrument used to measure certain quantities with a needle and a dial.



A cylindrical piece of soft iron core is free to rotate about a pivot P (top figure). A current runs through a wire (red) coiled around the core generating a magnetic field, $\vec{B}$, which tries to orient itself with the permanent background magnetic field (N-S) causing the core to rotate about P. A coiled spring (blue) attached to the pivot and an anchor point, A, resists the rotation of the core, and a balanced position is attained. By fixing a needle to the core, you have a device that can be calibrated to read current, voltage, or, as seen in the image below, a speedometer for a car. (Bonus points: can you name the year and make of the car to which this speedometer belongs? ⌣̈)

An important feature of a properly designed galvanometer is that it be critically damped. With insufficient damping, the needle bounces back and forth across the reading as the spring undergoes harmonic motion. With too much damping, the needle takes longer than necessary to settle on the reading.

a) Suppose the frequency of the undamped coiled spring is $f_0 = 5\,\mathrm{Hz}$ (bounces back and forth 5 times per second) and suppose further the galvanometer is critically damped. How long does it take for the needle to get to within 1% of its reading (the tolerance of a typical galvanometer)?

b) Now suppose a galvanometer with the same f_0 is overdamped, such that $\gamma = 2\omega_0$. How long does it take for the needle to get to within 1% of its reading now?

c) Repeat b) for an underdamped galvanometer where $\gamma = \omega_0/5$. How many periods of oscillation does the needle execute during this time?

For parts a and b, you should end up with a transcendental equation for t for which you'll need a root-finder to solve. This can be found on-line (*e.g.*, Wolfram-α), on your programmable calculator, or—gadzooks!—you could even write a short computer program!

Problem 20 Solve by "inspection", "direct integration", and/or "trial exponentials" the linear, second-order, inhomogeneous ODE:

$$y''(x) + 5y'(x) + 4y(x) \;=\; f(x), \tag{1}$$

for boundary conditions $y(0) = 1$ and $y'(0) = 0$, where:

a) $f(x) = 2$;

b) $f(x) = 5e^x$.

As discussed in §2.5 of the course notes, to solve such an *inhomogeneous* $[f(x) \neq 0]$ equation with boundary conditions, you must:

1. find two linearly independent solutions to the *homogeneous* equation [with $f(x) = 0$]; call these $y_1(x)$ and $y_2(x)$;

2. construct the general solution to the homogeneous equation,

$$y_\mathrm{h}(x) \;=\; Ay_1(x) + By_2(x),$$

   where A and B are constants;

3. by inspection or trial exponentials, find a *particular* solution, $y_\mathrm{p}(x)$ [*anything* that solves equation (1)];

4. write down the general solution to equation (1),

$$y(x) \;=\; y_\mathrm{h}(x) + y_\mathrm{p}(x);$$

5. and finally, apply boundary conditions to evaluate A and B.

Problem 21 (FC 3.10) A damped harmonic oscillator with $m = 10.0\,\mathrm{kg}$, $k = 250.\,\mathrm{N\,m^{-1}}$, and $b = 60.0\,\mathrm{kg\,s^{-1}}$ is subject to a driving force $F_0 \cos\omega t$, where $F_0 = 48.0\,\mathrm{N}$.

a) What value of ω results in steady-state oscillations with maximum amplitude (in the asymptotic limit)?

b) For the value of ω found in part a, what is the maximum amplitude and the corresponding phase shift?

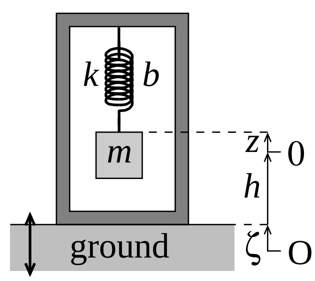
Problem 22 (FC 3.17) A damped harmonic oscillator is driven by an external force, $F(t) = F_0 \sin\omega t$. Show that the steady-state (particular) solution is given by,

$$x(t) = A(\omega)\sin(\omega t - \phi),$$

where $A(\omega)$ is given by equation (2.6.7) and ϕ is given by equation (2.6.4) in the class notes.

Problem 23 (Double problem) A simple seismometer can be constructed by assembling a mass, m, hanging from a damped spring, k and b, within a rigid structure as depicted in the figure. When a tremor occurs, the ground on which the seismograph is fixed rises up and down, driving the oscillator, m.

a) (4 points) Let h be the distance from the ground to the equilibrium position of the spring, let z be the distortion of the spring from equilibrium, and let ζ be the displacement of the ground relative to an inertial frame of reference, O, at rest relative to the average position of the seismometer.



Show that the differential equation of motion for m is:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = -\ddot{\zeta},$$

where, as usual, $\gamma = b/2m$ and $\omega_0^2 = k/m$.

b) (2 points) If we model the rising and falling of the ground as $\zeta(t) = \zeta_0 \cos\omega t$, find expressions for the phase and amplitude of the steady-state oscillations in terms of ω_0, ω, ζ_0, and γ.

c) (2 points) Using the expression $Q_\mathrm{d} = \omega_\mathrm{d}/2\gamma$ from the class notes, find γ in terms of ω_0 if $Q_\mathrm{d} = 2.00$.

d) (4 points) Find numerical values for A/ζ_0 and ϕ when the system is driven at:

$$i)\ \ \omega = \frac{\omega_0}{2}; \qquad\qquad ii)\ \ \omega = 2\omega_0.$$

Note that you don't need to know k or m and thus ω_0!

For phases, choose values for ϕ such that $0 < \phi < \pi$ rad. For what driving frequency, ω, is the displacement of m closer to being in phase and closer to being out of phase with the displacement of the ground?

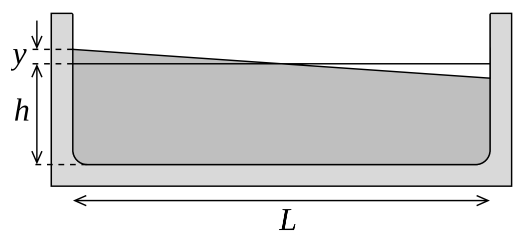
e) (5 points) From your expression for the amplitude, A, in part b, show that the resonant frequency and resonant amplitude are given by,

$$\omega_r = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 2\gamma^2}} \qquad \text{and} \qquad A_{max} = \frac{\omega_0^2 \zeta_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}.$$

Hint: If you aren't careful with the algebra, you can run into a mess pretty quickly once you try setting $dA/d\omega = 0$ to extremise A. You might find the algebra a little easier going if you try setting $d(A^2)/d(\omega^2) = 0$ instead.

f) (3 points) For $Q_d = 2$, find numerical values for ω_r/ω_0, the resonant amplitude, A_{max}/ζ_0, and the phase, ϕ_r, at resonance. Again, you don't need to know ω_0 for any of these values!

Problem 24 As depicted in the figure below, a bathtub of length L is filled with water to a depth h.



a) If the water is driven to slosh back and forth so that the water surface remains planar, show that the potential energy of the bathwater, U, is given by:

$$U(y) = \frac{\rho L w g}{6} y^2,$$

where ρ is the density of water, w is the width of the tub, and y is the additional depth of water above h at the left end of the tub ($x = 0$).

b) If you choose to do the bonus part d below, you'll find that the kinetic energy, K, of the horizontal motion of the water is given by,

$$K(\dot{y}) = \frac{\rho w L^3}{60h} \dot{y}^2.$$

Assuming the oscillation is weakly damped, set $E = U + K =$ constant, and show that the water level rises and falls as a simple harmonic oscillator. Thus, find its natural frequency, ω_0, and period of oscillation, T_0.

For small damping, $\omega_r \sim \omega_d \sim \omega_0$. Therefore, what is the resonant period of oscillation for a tub with $L = 1.5$ m and $h = 0.3$ m? Does this jibe with your intuition? (Surely *every* kid discovered that sliding back and forth along the bottom of the tub with just the "right" frequency will spill water onto the floor!)

c) The Bay of Fundy is about 250 km long. If we model it as a big bathtub sloshing back and forth as it is driven by solar and lunar tidal forces, the bay itself would represent half of the tub with the other half stuck in the Gulf of Maine. Thus, take $L \sim 500\,\text{km}$ and use $h \sim 50\,\text{m}$ as its effective depth. What is the resonant period of oscillation for the Bay of Fundy? Compare this with the driving frequency of the moon (two high tides per 24.9 hr, the time for the moon to return to the same spot in the sky), and in one sentence, explain why the Fundy tides are so notoriously high.

d) (5 bonus points) Show that the kinetic energy, K, of the horizontal motion of the water is given by:

$$K(\dot{y}) = \frac{\rho w L^3}{60h}\dot{y}^2.$$

Here, we assume the amplitude of oscillation is sufficiently small that only the back-and-forth motion of the water contributes to K, with the vertical motion contributing a negligible amount. And yet note that K as given is as a function of $\dot{y}$. Hmmm...

Hint: While part a should be a 3 or 4 liner, calculating K is a bit trickier and took me a full page to do so. The way I approached it was to calculate $v(x)$ (horizontal motion of the water) in terms of x, the horizontal position along the tub where $0 \le x \le L$, and $\dot{y}$, the rate at which the water level rises and falls at the left end of the tub. For this, I had to invoke the idea that the volume of water entering a fixed rectangle at location x of height h and width dx is the same as that leaving. Doing this, you should find $dv/dx = -\dot{z}/h$, where $\dot{z}$ is the speed at which water is rising or falling at point x. I then used a "similar triangles" argument to relate z and y. This is a tricky problem, so don't feel badly if you don't get it!

Problem 25 (FC 4.3) Find the constant c that makes the following forces conservative:

a) $\vec{F} = (xy, cx^2, z^3)$;

b) $\vec{F} = \left(\dfrac{z}{y}, \dfrac{cxz}{y^2}, \dfrac{x}{y}\right)$.

Problem 26 (FC 4.5) Let C_1 be the path on the x-y plane that joins $(0,0)$ and $(1,1)$ directly by the straight line $y = x$, and let C_2 be the path that joins $(0,0)$ to $(1,0)$ by the line $y = 0$, and then $(1,0)$ to $(1,1)$ by the line $x = 1$.

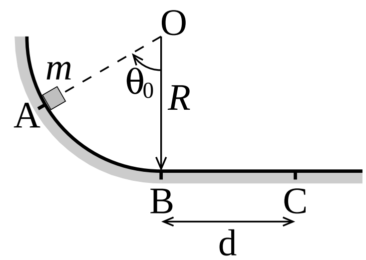
a) Evaluate $\int \vec{F} \cdot d\vec{r}$ for $\vec{F} = x\hat{\imath} + y\hat{\jmath}$ for each of the paths C_1 and C_2, and verify that $\vec{F}$ is conservative.

b) Evaluate $\int \vec{F} \cdot d\vec{r}$ for $\vec{F} = y\hat{\imath} - x\hat{\jmath}$ for each of the paths C_1 and C_2, and show that $\vec{F}$ is not conservative.

Problem 27 (FC 4.21) A smooth hemisphere of radius b is placed flat side down on a table. A mass m is placed on the hemisphere at a height $b/2$ above the table top. As it slides down the frictionless surface, at what height above the table top does m leave the surface of the hemisphere?

Problem 28 As shown in the figure, a track consists of a quarter circle of radius R attached seamlessly at point B to a straight portion of arbitrary length.

A mass, m, is placed on the curved portion at point A such that angle AOB $= \theta_0$, and released from rest. It slides down the track past point B, and stops at point C.



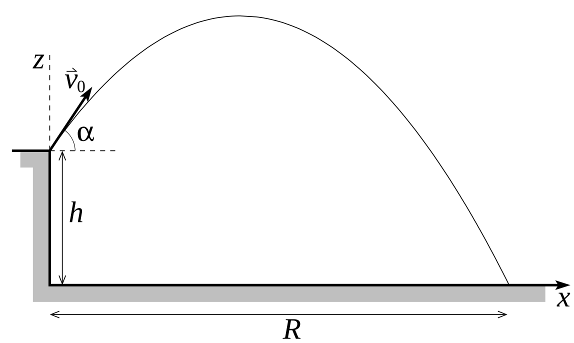
If the coefficient of kinetic friction is zero along AB and $\mu_\mathrm{k} > 0$ along BC, use the W-K theorem to find the distance d that m slides between B and C before coming to rest.

Problem 29 (FC 4.9) A cannon is situated at the edge of a bluff a height h over a plain below, and fires a cannon ball with a muzzle speed v_0.

a) Ignoring air resistance, show that the elevation angle, α, required to achieve maximum range across the plain is given by:

$$\sin^2\alpha = \frac{1}{2}\frac{v_0^2}{v_0^2 + gh}.$$

b) What is the maximum range, R_max, of the cannon in terms of v_0, h, and g?

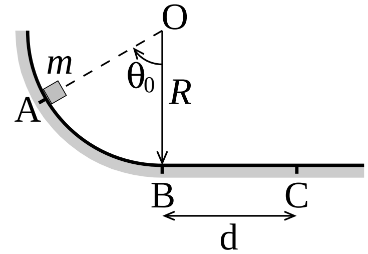


Problem 30 (Challenge problem, 10 point bonus) A track with coefficient of kinetic friction μ_k *over its entire length* consists of a quarter circle of radius R attached seamlessly at point B to a straight portion of arbitrary length. A mass, m, is placed on the curved portion at point A such that angle AOB $= \theta_0$, and released from rest. It slides down the track past point B, and comes to rest at point C, a distance d beyond B.

a) From Newton's 2nd Law, show that the centripetal acceleration, a_cp, of m as it slides along the quarter circle is given by the 1st order ODE:

$$\frac{da_\mathrm{cp}}{d\theta} - 2\mu_\mathrm{k} a_\mathrm{cp} = 2g(\mu_\mathrm{k}\cos\theta - \sin\theta), \qquad (1)$$

where $\theta_0 > \theta > 0$ is the angle mOB.



Hint: You may find it useful to know that the tangential acceleration and centripetal acceleration can be related as follows:

$$a_{\tan} = \frac{dv_{\tan}}{dt} = \frac{d\theta}{dt}\frac{dv_{\tan}}{d\theta} = -\frac{v_{\tan}}{R}\frac{dv_{\tan}}{d\theta} = -\frac{1}{2}\frac{d}{d\theta}\left(\frac{v_{\tan}^2}{R}\right) = -\frac{1}{2}\frac{da_{\rm cp}}{d\theta}.$$

Here, I've used $\dot{\theta} = -\dfrac{v_{\tan}}{R}$ since $v_{\tan}$ increases in the direction of decreasing θ.

b) Solve equation (1) for $a_{\rm cp}(\theta)$ by finding the homogeneous solution, adding on a particular solution, and then applying the boundary condition (at $\theta = \theta_0$, $a_{\rm cp} = 0$). After some algebra, show that at B ($\theta = 0$):

$$v_{\rm B}^2 = a_{\rm cp,B}R = \frac{2gR}{1+4\mu_{\rm k}^2}\Big[\big(1-2\mu_{\rm k}^2\big)\big(1-\cos\theta_0 e^{-2\mu_{\rm k}\theta_0}\big) - 3\mu_{\rm k}\sin\theta_0 e^{-2\mu_{\rm k}\theta_0}\Big] \qquad (2)$$

c) Use the W-K theorem between B and C to find the distance, d, where m comes to rest.

Problem 31 (FC 4.11 modified) A little leaguer hits a ball 0.75 m directly above home plate that attains a maximum height of 5.07 m and strikes the ground 25.0 m away.

a) Set up a coordinate system, (x, z), such that the trajectory of the ball, when extended behind the batter, intersects the ground at the origin, $(0, 0)$. Assuming no air resistance, show that the trajectory of the ball is given by:

$$3x^2 - 78x + 100z = 0,$$

with x and z measured in metres.

b) What is the horizontal distance of home plate from the origin?

c) Suppose an outfielder is capable of catching a ball that is anywhere between ground level and 2.07 m above the ground. Within what horizontal distance from home plate must the outfielder be to catch the ball?

By the way and for what it's worth, $5.07 = 3\left(\frac{13}{10}\right)^2$.

Problem 32 (FC 4.15) In class, we considered a projectile of mass m and velocity $\vec{v}$ experiencing a drag force given by $\vec{D} = -m\gamma\vec{v}$, where γ is the coefficient of linear air drag. By solving the differential equations stemming from Newton's 2$^{\rm nd}$ Law, we found the projectile *path* to be:

$$\vec{r}(t) = \frac{1}{\gamma}\left[\left(\vec{v}_0 + \frac{g}{\gamma}\hat{k}\right)\big(1-e^{-\gamma t}\big) - gt\hat{k}\right], \qquad (1)$$

where $\vec{v}_0$ is the projectile's initial velocity and $\hat{k}$ is a unit vector in the vertical (z) direction.

a) Show that the *trajectory* of the projectile is given by:

$$z(x) \;=\; \frac{x}{v_{0x}}\left(v_{0z}+\frac{g}{\gamma}\right)+\frac{g}{\gamma^2}\ln\left(1-\frac{\gamma x}{v_{0x}}\right), \tag{2}$$

where x is the horizontal direction along which the projectile travels, and $\vec{v}_0 = (v_{0x}, v_{0z})$.

b) Show that the range of the projectile, R (value of $x > 0$ when $z = 0$), is given by:

$$R \;=\; R_0\left(1-\frac{4\gamma v_0}{3g}\sin\alpha+\mathcal{O}(\gamma^2)\right),$$

where $R_0 = (v_0^2 \sin 2\alpha)/g$ is the range of the particle for $\gamma = 0$ (no air drag), α is the inclination angle of the projectile at the beginning of its trajectory, and $\mathcal{O}(\gamma^2)$ represents ignored terms of power γ^2 or higher.

Problem 33 (FC 4.16) The initial conditions of a 2-D isotropic oscillator are: $x(0) = A$, $y(0) = 4A$, $\dot{x}(0) = 0$, and $\dot{y}(0) = 3\omega A$, where ω is the angular frequency.

a) (7 points) Find the coordinates of the oscillator, (x, y), as functions of t, and show that the path is an ellipse entirely contained within a rectangle of dimensions $2A$ and $10A$.

b) (3 points) Find the inclination angle, ψ, of the semi-major axis of the ellipse relative to the $+x$ axis (equation 4.4.15 of the text), and make a sketch of the path.

Problem 34 (FC 4.17) A mass m is suspended by six massless springs aligned along the Cartesian axes (*e.g.*, Fig. 4.4.1 of eds. 6 and 7) with spring constants $(k_x, k_y, k_z) = k(1, 4, 9)$, where $k = \pi^2 m/2$ (and thus, with two springs in the x-direction, for example, $F_x = -2k_x x = -\pi^2 m x$).

a) If, at $t = 0$, $\vec{r} = \vec{0}$ and $\vec{v} = \dfrac{1}{\sqrt{3}}(1, 1, 1)$, find $\vec{r}(t)$.

b) Does m ever retrace its path and, if so, find the smallest value of t where m returns to its initial conditions.

Problem 35 (FC 4.20) An electron moves in the electromagnetic field $\vec{E} = E\hat{\jmath}$, $\vec{B} = B\hat{k}$, where E and B are constant. If $\vec{r}_0 = 0$ and $\vec{v}_0 = v_0\hat{\imath}$, show that the resulting particle path is a cycloid parameterised by t and given by:

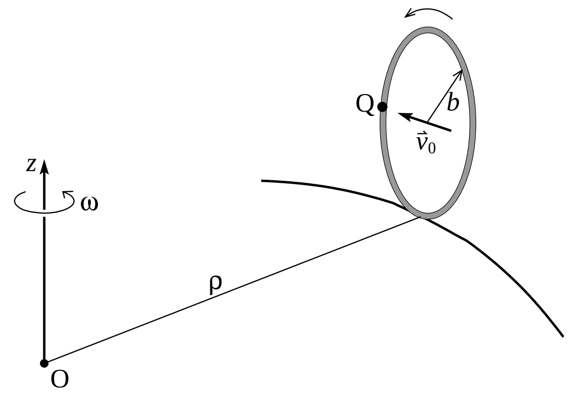
$$x(t) \;=\; a\sin\omega t + bt; \qquad y(t) \;=\; a(1-\cos\omega t); \qquad z(t) \;=\; 0,$$

where $\omega = eB/m$, $b = E/B$, and $a = (v_0 - b)/\omega$.

Problem 36 (FC 5.4) A plumb bob hangs from the ceiling of a boxcar in a train accelerating at $g/10$. If the plumb bob swings back and forth as a small-amplitude simple pendulum, find the period of oscillation.

Problem 37 (FC 5.9) A bicycle moves at constant speed v_0 and without slipping around a circular track of radius ρ (figure below). If the radius of the bicycle wheel is b, find the acceleration of the leading point, Q, on the wheel relative to the ground.

Guidance: This is the same problem as the example we did in class, except that there, we considered the acceleration of the point at the *top* of the wheel. Here, we are considering the leading point of the wheel. What I want you to do here is to go through the hunting and gathering process again (using the example we did in class as a template, if you like), being careful to note where any differences arise from choosing a different point on the wheel.



Problem 38 (FC 5.10) A bead of mass m is released from rest halfway along a smooth (frictionless) rod of length l rotating at a constant angular speed ω in the horizontal plane about one of its endpoints. (See example 5.3.3 in eds. 6 and 7.)

a) Find the displacement of the bead along the rod as a function of time.

b) When does the bead reach the end of the rod?

c) Find the velocity of the bead when it leaves the rod.

Problem 39 (FC 5.11) A rocket-boosted car of mass m heading due north reaches a speed of $v_0 = 648$ km/hr on the salt-flats of Utah (latitude $\lambda = 41^\circ$ N).

a) In terms of m, what is the Coriolis force acting on the car?

b) What is the numerical value of the ratio between the magnitude of the Coriolis force and the weight of the car?

Problem 40 A bullet is fired from a rifle with speed v_0' due east at a northern latitude λ and at an elevation angle α (angle between the rifle's long axis and the horizontal). Neglecting both air resistance and the variation of gravity with height, this problem shows how the earth's rotation affects the range of a rifle.

a) (3 points) First, show that the time it takes for the bullet to strike the ground is,

$$t = \frac{2v_0' \sin\alpha}{g}\left(1 - \frac{2\omega v_0'}{g}\cos\alpha\cos\lambda\right)^{-1} \tag{1}$$

where ω is the angular speed of the earth rotating on its axis.

b) (2 points) Assuming $\omega v_0' \ll g$, use a binomial expansion keeping just the first two terms to show that equation (1) may be written approximately as,

$$t \approx \frac{2v_0' \sin\alpha}{g} + \frac{2\omega R_0 \cos\lambda}{g},$$

where $R_0 = \dfrac{v_0'^2 \sin 2\alpha}{g}$ is the rifle range without taking the earth's rotation into account.

c) (5 points) Neglecting any north-south deviation, show that the rifle range is given by,

$$R_0' = R_0 + \sqrt{\frac{2R_0^3}{g}}\,\omega\cos\lambda\left(\cot^{1/2}\alpha - \frac{1}{3}\tan^{3/2}\alpha\right),$$

where all terms proportional to ω^2 or a higher power have been discarded. On the same token, *all* terms proportional to ω to the first power have been retained!