Resonance of a driven oscillator

PHYS 2302, Saint Mary's University D. A. Clarke, July 2019

In class, we found the amplitude and phase of a damped harmonic oscillator driven by a force $F(t) = F_0 \cos \omega t$ are given by:

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}; \tag{1}$$

$$\phi(\omega) = \tan^{-1} \frac{2\gamma\omega}{\omega_0^2 - \omega^2}. \tag{2}$$

where: m = mass of oscillator;

ω = driving frequency;

$\omega_0 = \sqrt{\frac{k}{m}}$ = oscillation frequency of undamped system;

$\gamma = \frac{b}{2m}$ = damping coefficient.

Also shown in class, $A(\omega)$ is a maximum at the *resonant frequency*,

$$\omega_\mathrm{r} = \sqrt{\omega_0^2 - 2\gamma^2},$$

which is real so long as $\gamma < \frac{\omega_0}{\sqrt{2}}$. For $\omega = \omega_\mathrm{r}$, Eq. (1) becomes:

$$A_\mathrm{max} = \frac{F_0}{m\sqrt{\omega_0^4 - \omega_\mathrm{r}^4}}. \tag{3}$$

Before plotting equations (1) – (3), we first scale them. Let $\alpha = \gamma/\omega_0$ and $\beta = \omega/\omega_0$, both unitless. Then,

$$A = \frac{F_0}{\underbrace{m\omega_0^2}_{k}\sqrt{(1-\beta^2)^2 + 4\alpha^2\beta^2}}$$

$$\Rightarrow \quad \tilde{A}(\beta) \equiv \frac{Ak}{F_0} = \frac{1}{\sqrt{(1-\beta^2)^2 + 4\alpha^2\beta^2}}, \tag{4}$$

where $\tilde{A}$ is also a unitless quantity.

Next,

$$\tan\phi(\beta) = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}\,\frac{\omega_0^2}{\omega_0^2} = \frac{2\alpha\beta}{1-\beta^2}. \tag{5}$$

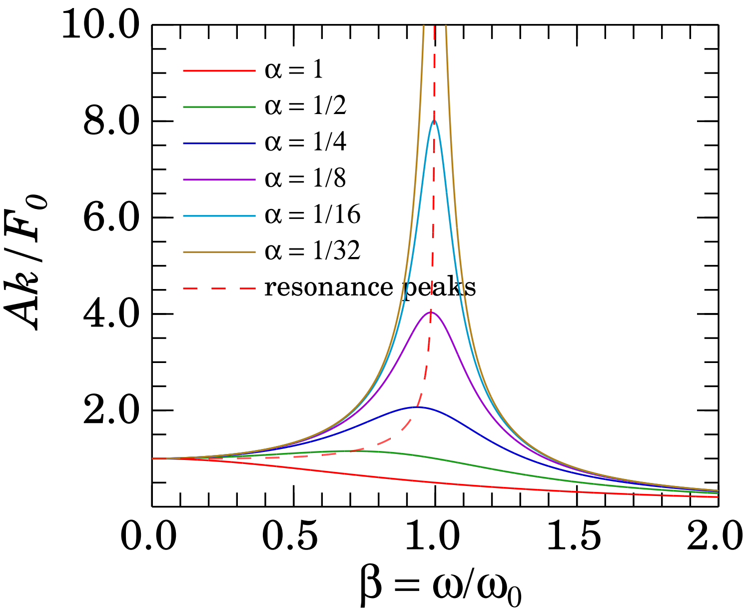
Finally, for $\beta_{\rm r} \equiv \omega_{\rm r}/\omega_0$,

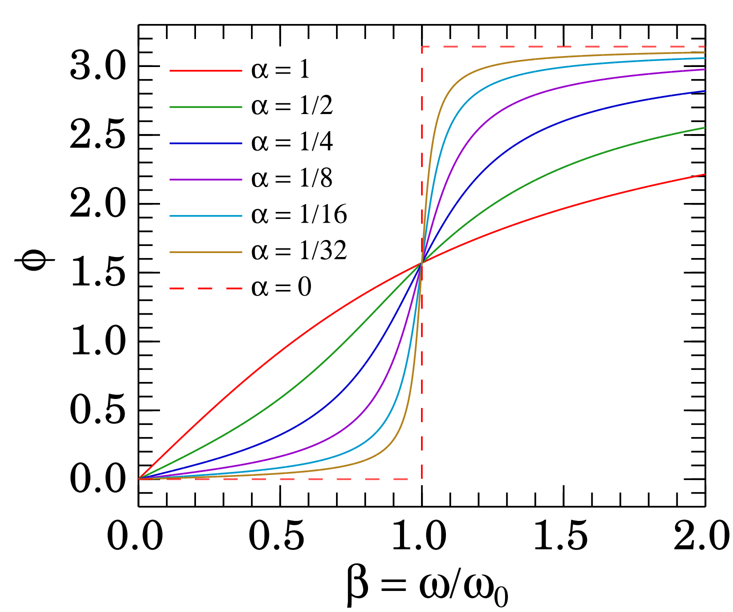
$$A_{\rm max} = \frac{F_0}{m\omega_0^2\sqrt{1-\beta_{\rm r}^4}} \quad \Rightarrow \quad \tilde{A}_{\rm max} = \frac{A_{\rm max}k}{F_0} = \frac{1}{\sqrt{1-\beta_{\rm r}^4}}, \tag{6}$$

for $0 < \beta_{\rm r} < 1$.

Equations (4) – (6) can now be plotted by specifying:

- a *domain* for β [*e.g.*, $\in (0:2)$, where $\beta = 1 \Rightarrow$ system is driven at the undamped oscillation frequency];
- α, unitless damping coefficient, γ/ω_0;
- β, unitless driving frequency, ω/ω_0.





Left: Unitless amplitude, $\tilde{A}$, plotted against unitless driving frequency, β, for various values of unitless damping coefficient, α [Eq. (3)]. The peak amplitude for each α occurs at the unitless resonance frequency,

$$\beta_{\rm r} = \frac{\omega_{\rm r}}{\omega_0} = \frac{1}{\omega_0}\sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{1-2\alpha^2}.$$

The dashed red line is $\tilde{A}_{\max}$ plotted against $\beta = \beta_{\rm r} \in (0,1)$ [Eq. (6)]. Since $\beta_{\rm r} \in \mathbb{R}$, there can be no resonant frequency for $\alpha > \frac{1}{\sqrt{2}} \sim 0.707$. Thus, there is an $A_{\max}$ (barely) for $\alpha = 0.5$ (green), but not for $\alpha = 1$ (solid red) where the plot decreases monotonically from $\beta = 0$.

Right: Phase lag, ϕ, plotted against β for various α [Eq. (5)]. As $\alpha \to 0$ (no damping), $\phi(\beta)$ becomes a step function (dashed red line), with $x(t)$ and $F(t)$ in phase ($\phi = 0$) for $\omega < \omega_0$, and of opposite phase ($\phi = \pi$) for $\omega > \omega_0$.

A lesson on the importance of understanding resonance: *Galloping Gertie*, the Tacoma Narrows Bridge built in 1940.