Solutions to Assignment 1

PHYS 2302 (Mechanics I); D. A. Clarke

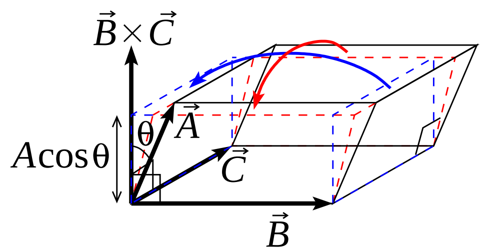
Problem 1 (FC 1.12)

a) Show that the volume of the parallelepiped whose sides are made up of vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ (none of which need be perpendicular to each other) is $|\vec{A}\cdot(\vec{B}\times\vec{C})|$.

b) Explain why this proves the "cyclic rule", namely:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A}).$$

Solution: a) From problem 1.10, the area of the base of the parallelepiped (itself a parallelogram) is $|\vec{B}\times\vec{C}|$.



Further, the vector $\vec{B}\times\vec{C}$ is $\perp$ to the plane defined by $\vec{B}$ and $\vec{C}$.

Now, using an argument similar to that in problem 1.10, we cut away "prisms" from one side of the parallelepiped (*e.g.*, red and blue-dashed portions) and move them to the opposite side, thereby "squaring up" the sides. The resulting object has a parallelogram base with area $|\vec{B}\times\vec{C}|$ and vertical sides with height $A\cos\theta$ (where θ is the angle between $\vec{A}$ and $\vec{B}\times\vec{C}$). The volume of such an object is then its height times its base:

$$V = A\cos\theta|\vec{B}\times\vec{C}| = |\vec{A}\cdot(\vec{B}\times\vec{C})|,$$

as desired.

b) We could equally have chosen the side defined by vectors $\vec{A}$ and $\vec{B}$ to be the "base", in which case we would have found the volume of the parallelepiped to be $|\vec{C}\cdot(\vec{A}\times\vec{B})|$. Similarly, if we chose $\vec{A}$ and $\vec{C}$ to define the "base", $|\vec{B}\cdot(\vec{C}\times\vec{A})|$ would have been the volume. Since all expressions for the volume must be the same, we have the "cyclic rule", namely:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A}).$$

Problem 2 (FC 1.22) An ant crawls on the surface of a ball of radius b such that the ant's motion in spherical polar coordinates is given by:

$$r = b; \qquad \theta = \frac{\pi}{2} + \frac{\pi}{8}\cos(4\omega t); \qquad \phi = \omega t, \tag{1}$$

where ω is constant.

a) Find the speed of the ant as a function of time.

b) Describe the path represented by equations (1). In particular, how many periods of oscillation does the ant undertake per circumnavigation of the sphere?

Solution: a) Using equation (1.12.12) in the text, we have:

$$\vec{v} = \overset{0}{\cancel{\dot{r}}}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\sin\theta\hat{e}_\phi = -b\omega\frac{\pi}{2}\sin(4\omega t)\hat{e}_\theta + b\omega\sin\left(\frac{\pi}{2}+\frac{\pi}{8}\cos(4\omega t)\right)\hat{e}_\phi$$

$$= -b\omega\frac{\pi}{2}\sin(4\omega t)\hat{e}_\theta + b\omega\cos\left(\frac{\pi}{8}\cos(4\omega t)\right)\hat{e}_\phi$$

$$\Rightarrow \quad \boxed{v = |\vec{v}| = b\omega\sqrt{\frac{\pi^2}{4}\sin^2(4\omega t) + \cos^2\left(\frac{\pi}{8}\cos(4\omega t)\right)}.}$$

b) Without the cosine term, $\theta = \pi/2$ and the ant's path is a continuous circle about the equator where each circumnavigation takes $t = 2\pi/\omega$ in time.

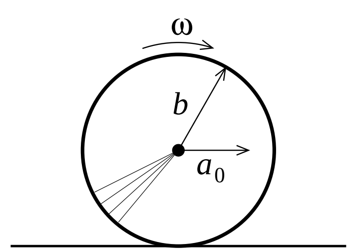
The cosine term adds a sinusoidal variation of θ about $\pi/2$ (and thus the equator) with an amplitude of $\pi/8$ ($5\pi/8 \le \theta \le 3\pi/8$) and a period $T = 2\pi/(4\omega) = \pi/(2\omega)$. Since it takes the ant $t = 2\pi/\omega$ to circumnavigate the sphere, there are $t/T = 4$ full sinusoidal periods per circumnavigation.

Problem 3 (FC 1.28) A wheel of radius b rolls along the ground at constant forward acceleration, a_0. As a function of time, find $|\vec{a}\,|$ of any point on the rim of the wheel:

a) relative to the centre of the wheel; and

b) relative to a point on the ground.

c) At a given time, t, which point on the rim of the wheel has the greatest acceleration relative to the ground? Where is this point as $t \to \infty$?



Note: to specify a point along the rim of the wheel, it is sufficient to specify the angular coordinate, θ, counterclockwise from the $+x$-axis (the direction of $\vec{a}_0$ in the figure).

Solution: a) Let v be the speed of the wheel axis relative to the ground at any given time, and assume the wheel rolls without slipping. Then, in the frame of reference of the wheel axis,

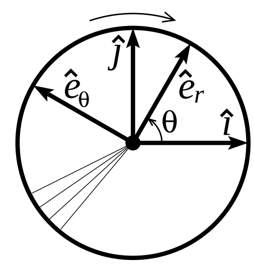
$$\dot{\theta} = -\omega = -\frac{v}{b} = -\frac{a_0 t}{b} \qquad \text{and} \qquad \ddot{\theta} = -\frac{a_0}{b}, \tag{1}$$

where a negative $\dot{\theta}$ and $\ddot{\theta}$ mean they are both in the clockwise sense. Using 2-D polar coordinates, substitute equations (1) and $\dot{r} = \ddot{r} = 0$ into equation 1.11.9 in eds. 6 and 7 to get:

$$\vec{a} \;=\; -b\left(-\frac{a_0 t}{b}\right)^2 \hat{e}_r + b\left(-\frac{a_0}{b}\right)\hat{e}_\theta \;=\; -\frac{(a_0 t)^2}{b}\hat{e}_r - a_0\hat{e}_\theta \qquad (2)$$

$$\Rightarrow \quad \boxed{|\vec{a}\,| \;=\; \sqrt{\frac{(a_0 t)^4}{b^2} + {a_0}^2} \;=\; a_0\sqrt{\frac{a_0^2 t^4}{b^2} + 1}.}$$

b) Orienting the Cartesian unit vectors $\hat{\imath}$ and $\hat{\jmath}$ in the usual way with θ measured counterclockwise with respect to $\hat{\imath}$ (as shown in the figure), the axle of the wheel accelerates at $a_0\hat{\imath}$ relative to the ground. Thus, the acceleration of a point on the rim of the wheel relative to the ground is:



$$\vec{a}_{\rm g} \;=\; \vec{a} + a_0\hat{\imath} \;=\; -\frac{v^2}{b}\hat{e}_r - a_0\hat{e}_\theta + a_0\hat{\imath}, \qquad (3)$$

using equation (2) and $v = a_0 t$. But, from the diagram,

$$\hat{e}_r \;=\; \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \qquad \text{and} \qquad \hat{e}_\theta \;=\; -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta.$$

Substitute these into equation (3) to get:

$$\begin{aligned}
\vec{a}_{\rm g} \;&=\; -\frac{v^2}{b}(\hat{\imath}\cos\theta + \hat{\jmath}\sin\theta) - a_0(-\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta) + a_0\hat{\imath} \\
&=\; \hat{\imath}\left(-\frac{v^2}{b}\cos\theta + a_0\sin\theta + a_0\right) - \hat{\jmath}\left(\frac{v^2}{b}\sin\theta + a_0\cos\theta\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad |\vec{a}_{\rm g}| \;&=\; \sqrt{\left(-\frac{v^2}{b}\cos\theta + a_0\sin\theta + a_0\right)^2 + \left(\frac{v^2}{b}\sin\theta + a_0\cos\theta\right)^2} \\
&=\; \sqrt{\frac{v^4}{b^2}\cos^2\theta + a_0^2\sin^2\theta + a_0^2 - \cancel{2\frac{v^2 a_0}{b}\cos\theta\sin\theta} - 2\frac{v^2 a_0}{b}\cos\theta \;\ldots} \\
&\qquad\qquad \overline{\ldots + 2a_0^2\sin\theta + \frac{v^4}{b^2}\sin^2\theta + a_0^2\cos^2\theta + \cancel{2\frac{v^2 a_0}{b}\sin\theta\cos\theta}} \\
&=\; \sqrt{\frac{v^4}{b^2} + 2a_0^2 - 2\frac{v^2 a_0}{b}\cos\theta + 2{a_0}^2\sin\theta}
\end{aligned}$$

$$\Rightarrow \quad \boxed{|\vec{a}_{\rm g}| \;=\; a_0\sqrt{2 + 2\sin\theta + \frac{a_0^2 t^4}{b^2} - 2\frac{a_0 t^2}{b}\cos\theta}.} \qquad (4)$$

c) Differentiating equation (4) with respect to θ and setting the result to zero, we get:

$$\frac{d|\vec{a}_{\rm g}|}{d\theta} \;=\; \frac{a_0}{2\sqrt{\sim}}\left(2\cos\theta + 2\frac{a_0 t^2}{b}\sin\theta\right) \;=\; 0 \quad \Rightarrow \quad \tan\theta \;=\; -\frac{b}{a_0 t^2}$$

$$\Rightarrow \quad \boxed{\theta = n\pi - \tan^{-1}\left(\frac{b}{a_0 t^2}\right),\ n \in \mathbb{Z}.}$$

Case 1: $n = 1$ (or any odd n)

$\theta = \pi - \tan^{-1}(b/a_0 t^2)$. Thus, for $t = 0$, $\theta = \pi/2$ and as $\boxed{t \to \infty,\ \theta \to \pi.}$

Thus, $n = 1$ corresponds to $\pi < \theta \leq \pi/2$, the upper left (trailing) quarter of the wheel, and it is here where $|\vec{a}_g|$ is a *maximum*. As $t \to \infty$, $v \to \infty$ as well, and the centripetal acceleration is the dominant term. At that point, the maximum $|\vec{a}_g|$ is where the centripetal acceleration points in the same direction as the translational acceleration of the wheel—namely $\hat{\imath}$—which occurs at $\theta = \pi$.

Case 2: $n = 0$ (or any even n)

$\theta = -\tan^{-1}(b/a_0 t^2)$. Thus, for $t = 0$, $\theta = -\pi/2$. Then as $t \to \infty$, $\theta \to 0$.

Thus, $n = 0$ corresponds to $-\pi/2 \leq \theta < 0$, points diametrically opposite to those in case 1 and thus in the lower right (leading) quarter of the wheel. This is where $|\vec{a}_g|$ is a *minimum*.

Problem 4 A mass m rests on top of a larger mass, M, which in turn rests on a frictionless surface. When M is prevented from moving, as shown in Fig. 1, F_0 is the maximum horizontal force that may be applied directly to m before m starts slipping along the top of M.

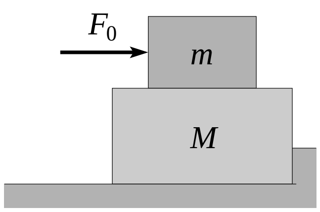


Fig. 1. F_0 applied to m

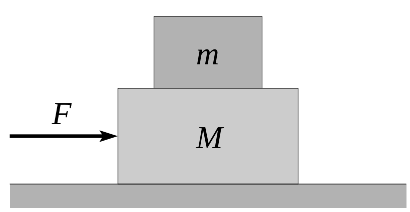


Fig. 2. F applied to M

a) In Fig. 2, the barrier preventing M from moving is removed and a horizontal force F is applied directly to M. In terms of m, M, and F_0, find the maximum value of F, F_{max}, that may be applied without causing m to slip, and the corresponding acceleration of the two masses, a.

b) If $m = 4.00$ kg, $M = 5.00$ kg, and $F_0 = 12.0$ N, find numerical values for F_{max} and a.

Solution:

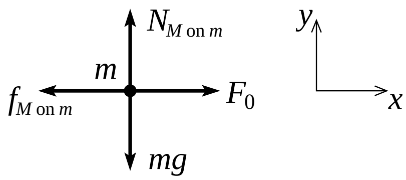


Fig. 3. FBD for Fig. 1.

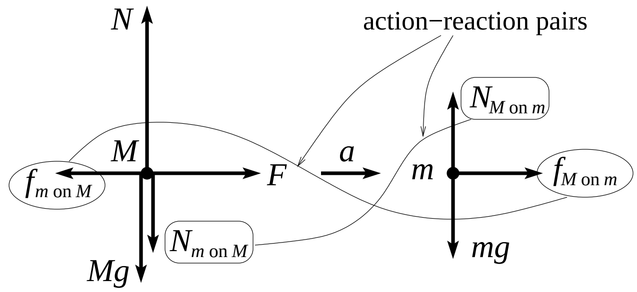


Fig. 4. FBDs for Fig. 2.

The free-body diagram corresponding to Fig. 1 is shown in Fig. 3, from which we may write:

$$\sum F_x = ma_x \quad \Rightarrow \quad F_0 - f_{M\,\text{on}\,m} = 0 \quad \Rightarrow \quad f_{M\,\text{on}\,m} = F_0. \tag{1}$$

The subscript "M on m" indicates that the force is exerted by mass M on mass m which, in this case, applies to both normal and frictional forces.

Figure 4 shows the FBDs corresponding to Fig. 2 and for which Newton's 2$^{\text{nd}}$ Law gives us:

$$\text{for } M: \quad x/ \ \sum F_x = Ma_x \quad \Rightarrow \quad F - f_{m\,\text{on}\,M} = Ma; \tag{2}$$

$$y/ \ \sum F_y = Ma_y \quad \Rightarrow \quad N - Mg - N_{m\,\text{on}\,M} = 0; \tag{3}$$

$$\text{for } m: \quad x/ \ \sum F_x = ma_x \quad \Rightarrow \quad f_{M\,\text{on}\,m} = ma; \tag{4}$$

$$y/ \ \sum F_y = ma_y \quad \Rightarrow \quad N_{M\,\text{on}\,m} - mg = 0. \tag{5}$$

Equations (3) and (5) are not relevant to the problem, and are discarded.

The pair of forces $\vec{f}_{m\,\text{on}\,M}$ and $\vec{f}_{M\,\text{on}\,m}$ is an action-reaction pair, and thus their magnitudes are equal:

$$f_{m\,\text{on}\,M} = f_{M\,\text{on}\,m} = F_0, \tag{6}$$

from equation (1). Note that the fact these forces point in opposite directions is taken into account by the different directions of their representative arrows in the FBD (Fig. 4), and we do *not* introduce a negative sign in equation (6) which describes only the *magnitudes*.

Substituting equation (6) into equations (4) and (2), we get:

$$F_0 = ma \quad \Rightarrow \quad \boxed{a = \frac{F_0}{m};}$$

$$F_{\text{max}} - F_0 = Ma = M\frac{F_0}{m} \quad \Rightarrow \quad \boxed{F_{\text{max}} = F_0\left(1 + \frac{M}{m}\right).}$$

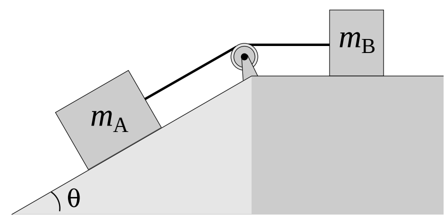
Note that even though static friction was involved in the solution, nowhere did we need to know anything about μ_s, nor that $f_s = \mu_s N$.

c) If $m = 4.00$ kg, $M = 5.00$ kg, and $F_0 = 12.0$ N, then $\boxed{a = 3.00\ \text{m s}^{-2} \text{ and } F_{\text{max}} = 27.0\ \text{N.}}$

Problem 5 The figure shows a frictionless inclined surface joining a horizontal surface with a kinetic coefficient of friction μ_k between it and the mass m_B. Let the cord attaching m_A and m_B be massless, and let the pulley be massless and frictionless.

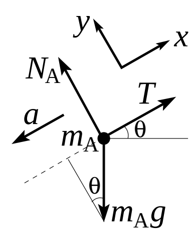
a) Find the tension in the cord and the acceleration of the blocks.

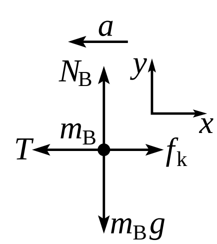
b) If $m_A = 4.00$ kg, $m_B = 2.00$ kg, $\theta = 30°$, and $\mu_k = 0.5$, find numerical values for T and a.



Solution: a) Using the FBDs for the two masses, we apply Newton's 2nd law, $\sum \vec{F} = m\vec{a}$, to get,

$$\begin{aligned}
m_A:\quad x/ \qquad T - m_A g \sin\theta &= -m_A a; && (1)\\
y/ \qquad N_A - m_A g \cos\theta &= 0, && (2)\\
m_B:\quad x/ \qquad f_k - T &= -m_B a; && (3)\\
y/ \qquad N_B - m_B g &= 0. && (4)
\end{aligned}$$





To find a, add equations (1) and (3):

$$f_k - m_A g \sin\theta = -(m_A + m_B)a \quad \Rightarrow \quad a = \frac{m_A g \sin\theta - f_k}{m_A + m_B}. \tag{5}$$

But, $f_k = \mu_k N_B$ and, from (4), $N_B = m_B g$. Thus, $f_k = \mu_k m_B g$, and (5) becomes:

$$\boxed{a = g\frac{m_A \sin\theta - m_B \mu_k}{m_A + m_B}.} \tag{6}$$

To find T, substitute (6) into (1):

$$\begin{aligned}
T &= m_A(g\sin\theta - a) = m_A g\left(\sin\theta - \frac{m_A \sin\theta - m_B\mu_k}{m_A + m_B}\right)\\
&= g\frac{m_A}{m_A + m_B}\left(\cancel{m_A \sin\theta} + m_B \sin\theta - \cancel{m_A \sin\theta} + m_B\mu_k\right)
\end{aligned}$$

$$\Rightarrow \quad \boxed{T = g\frac{m_A m_B}{m_A + m_B}(\sin\theta + \mu_k).} \tag{7}$$

b) Using $m_A = 4.00$ kg, $m_B = 2.00$ kg, $\theta = 30°$, and $\mu_k = 0.5$ in equations (6) and (7), we get:

$$a = \frac{g}{6} \sim 1.64\,\mathrm{m\,s^{-2}} \quad \text{and} \quad T = g\frac{8}{6}(1)\,\mathrm{kg} \sim 13.1\,\mathrm{N}.$$