Solutions to Assignment 2

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1 For each of the following first order ODEs, find $y(x)$ using separation of variables:

a) $(x+1)\dfrac{dy}{dx} - y \;=\; 1;$

b) $(x^2-1)\cot y\dfrac{dy}{dx} \;=\; 1;$

c) $x\dfrac{dy}{dx} - y - x^2(1-x) \;=\; 0.$

Hint: Turns out that as written, the ODE in part c is not separable. However, if you rearrange the equation as:

$$\frac{xy'-y}{x^2} \;=\; 1-x,$$

you might recognise $(xy'-y)/x^2 \;=\; (y/x)'$ as a *perfect derivative*. Hmmmm, this might suggest a substitution of, say, $z = y/x$...

Solution: a) Separating the variables, we get:

$$\frac{dy}{dx} = \frac{y+1}{x+1} \quad\Rightarrow\quad \frac{dy}{y+1} \;=\; \frac{dx}{x+1},$$

which can now be integrated,

$$\int\frac{dy}{y+1} \;=\; \ln(y+1) \;=\; \int\frac{dx}{x+1} \;=\; \ln(x+1)+c,$$

where c is the combined constant of integration. Solving for y, we get:

$$e^{\ln(y+1)} \;=\; y+1 \;=\; e^{\ln(x+1)+c} \;=\; e^c e^{\ln(x+1)} \;=\; C(x+1) \quad\Rightarrow\quad \boxed{y(x) \;=\; C(x+1)-1,}$$

where $C = e^c$ is a "constant of integration".

b) Separating the variables, we get:

$$\frac{\cos y}{\sin y}\,dy \;=\; \frac{dx}{x^2-1} \;=\; \frac{1}{2}\frac{dx}{x-1} - \frac{1}{2}\frac{dx}{x+1},$$

using partial fractions. Letting $z = \sin y$ (and thus $dz = \cos y\,dy$) and integrating, we get,

$$\int\frac{dz}{z} \;=\; \ln z \;=\; \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1} \;=\; \frac{1}{2}\big(\ln(x-1)-\ln(x+1)\big)+c$$

$$= \ln\left(\frac{x-1}{x+1}\right)^{1/2} + c,$$

where c is the combined constant of integration. Thus,

$$e^{\ln z} = z = \sin y = e^c\sqrt{\frac{x-1}{x+1}} \quad\Rightarrow\quad \boxed{y(x) = \sin^{-1}\left(C\sqrt{\frac{x-1}{x+1}}\right),}$$

where $C = e^c$ is the "constant of integration".

c) Following the hint, rewrite the ODE as,

$$\frac{xy' - y}{x^2} = \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{dz}{dx} = 1 - x,$$

where $z = y/x$. In terms of z, this ODE *is* separable, and we write:

$$dz = (1-x)dx \quad\Rightarrow\quad z = \frac{y}{x} = x - \frac{x^2}{2} + c,$$

where c is the combined constant of integration. Solving for y, we get:

$$\boxed{y(x) = cx + x^2 - \frac{x^3}{2}.}$$

If you keep open to the possibility of a variable substitution such as in part c, you'll find many first order ODEs are separable, even if they don't look like they are at first blush.

Problem 2 (FC 2.1) Find $v(t)$ and $x(t)$ for a mass, m, starting from rest at $x = 0$ and $t = 0$, subject to the forces:

a) $F_x = F_0 + ct$;

b) $F_0 \sin(ct)$; and

c) $F_0 e^{ct}$.

Solution: First, note that for forces that are functions only of t, and where at $t = 0$, both $v = 0$ and $x = 0$, we can write:

$$a(t) = \frac{dv}{dt} \quad\Rightarrow\quad dv = a(t)dt \quad\Rightarrow\quad \int_0^{v(t)} dv = v(t) = \int_0^t a(t')dt'$$

$$\text{and}\quad v(t) = \frac{dx}{dt} \quad\Rightarrow\quad dx = v(t)dt \quad\Rightarrow\quad \int_0^{x(t)} dx = x(t) = \int_0^t v(t')dt',$$

where $a(t) = F_x(t)/m$.

a) $F_x = F_0 + ct \quad \Rightarrow$

$$\begin{aligned} v(t) &= \int_0^t a(t')dt' = \frac{1}{m}\int_0^t (F_0 + ct')dt' = \frac{1}{m}\left(F_0 t + \tfrac{1}{2}ct^2\right) \\ x(t) &= \int_0^t v(t')dt' = \frac{1}{m}\int_0^t \left(F_0 t' + \tfrac{1}{2}ct'^2\right) dt' = \frac{1}{m}\left(\tfrac{1}{2}F_0 t^2 + \tfrac{1}{6}ct^3\right). \end{aligned}$$

b) $F_x = F_0 \sin(ct) \quad \Rightarrow$

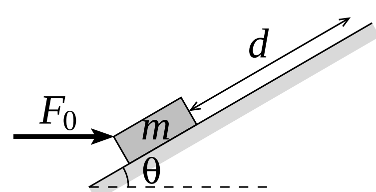
$$\begin{aligned} v(t) &= \int_0^t a(t')dt' = \frac{F_0}{m}\int_0^t \sin(ct')dt' = -\frac{F_0}{mc}\cos(ct')\Big|_0^t = \frac{F_0}{mc}\big(1-\cos(ct)\big) \\ x(t) &= \int_0^t v(t')dt' = \frac{F_0}{mc}\int_0^t \big(1-\cos(ct')\big)dt' = \frac{F_0}{mc^2}\big(ct' - \sin(ct')\big)\Big|_0^t = \frac{F_0}{mc^2}\big(ct - \sin(ct)\big). \end{aligned}$$

c) $F_x = F_0 e^{ct} \quad \Rightarrow$

$$\begin{aligned} v(t) &= \int_0^t a(t')dt' = \frac{F_0}{m}\int_0^t e^{ct'}dt' = \frac{F_0}{mc}e^{ct'}\Big|_0^t = \frac{F_0}{mc}\big(e^{ct}-1\big) \\ x(t) &= \int_0^t v(t')dt' = \frac{F_0}{mc}\int_0^t \big(e^{ct'}-1\big)dt' = \frac{F_0}{mc^2}\big(e^{ct'} - ct'\big)\Big|_0^t = \frac{F_0}{mc^2}\big(e^{ct} - ct - 1\big). \end{aligned}$$

Problem 3 Starting from rest, a block of mass m is pushed up a ramp with inclination angle θ by a force F_0 applied horizontally. After m moves up the ramp a distance d, its speed is v upward along the ramp.

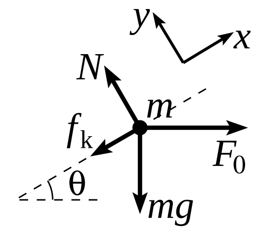
Using the work-kinetic theorem and the following data: $m = 1.00\,\mathrm{kg}$; $g = 9.81\,\mathrm{m\,s^{-2}}$; $F_0 = 12.0\,\mathrm{N}$; $\theta = 30°$; $d = 0.270\,\mathrm{m}$; and $v = 1.26\,\mathrm{m\,s^{-1}}$, find a numerical value (to three significant figures) for the coefficient of kinetic friction, μ_k, between the block and the ramp surface.



Hint: As derived in class so far, the W-K theorem is for rectilinear (1-D) motion. Thus, if the displacement is along the x axis and the x-component of a constant force $\vec{F}$ is F_x, the work done by $\vec{F}$ along $0 \le x \le d$ is given by:

$$W_F = \int_0^d F_x\,dx = F_x\int_0^d dx = F_x d. \tag{1}$$

Solution: From the FBD, the forces are F_0, mg, N, and f_k, each of which is constant. Thus, from the work-kinetic theorem and equation (1),



$$
\begin{aligned}
\sum W &= \text{work done by all external forces} = \Delta K \\
&= W_{F_0} + W_{mg} + W_N + W_{f_k} \\
&= (F_0)_x d + (mg)_x d + N_x d + (f_k)_x d \\
&= (F_0 \cos\theta)d + (-mg\sin\theta)d + (0)d + (-\mu_k N)d \\
&= \big(F_0\cos\theta - mg\sin\theta - \mu_k(mg\cos\theta + F_0\sin\theta)\big)d = K_\text{f} - \cancelto{0}{K_\text{i}} = \tfrac{1}{2}mv^2,
\end{aligned}
$$

since, from the FBD in the y-direction, $N = mg\cos\theta + F_0\sin\theta$. Solving for μ_k, we get:

$$\mu_\text{k} = \frac{F_0\cos\theta - mg\sin\theta - \frac{1}{2}mv^2/d}{mg\cos\theta + F_0\sin\theta},$$

and plugging in numbers, we get,

$$\mu_\text{k} = \frac{(12.0)\sqrt{3}/2 - (1.00)(9.81)/2 - \frac{1}{2}(1.00)(1.26)^2/(0.270)}{(1.00)(9.81)\sqrt{3}/2 + (12.0)/2} \sim \underline{\underline{0.176}}.$$

Problem 4 (FC 2.8, 2.15)

a) A mass, m, moves with velocity $v(x) = b/x^3$, where $b > 0$ is a constant. Find $F(x)$ acting on the mass.

b) Taking into account both the linear and quadratic terms in the air drag force,

$$D = -c_1 v - c_2 v^2, \tag{1}$$

(*e.g.*, equation 2.4.3, p. 69, ed. 7 of F&C), show that the terminal speed of a falling object of mass m is given by:

$$v_\text{t} = \sqrt{\frac{mg}{c_2} + \left(\frac{c_1}{2c_2}\right)^2} - \frac{c_1}{2c_2}.$$

Solution: a) Since we want $F(x)$ from $v(x)$, we want a version of Newton's second law in terms of v and x rather than x and t. Thus,

$$F(x) = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = m\frac{dx}{dt}\frac{dv}{dx} = mv\frac{dv}{dx}.$$

Substituting $v(x) = b/x^3$, we get:

$$F(x) = m\frac{b}{x^3}\frac{d}{dx}\left(\frac{b}{x^3}\right) = -3m\frac{b}{x^3}\frac{b}{x^4} \quad\Rightarrow\quad \boxed{F(x) = -\frac{3mb^2}{x^7}.}$$

b) This is an example of a problem that may look tricky at first blush, but where the solution is embarrassingly simple. No differential equations to solve, unless you really want practise doing messy integrals ...

For an object of mass m in free fall with both gravity and the air drag of equation (1) acting on it, Newton's second law states:

$$\sum F = ma \quad \Rightarrow \quad mg - c_1 v - c_2 v^2 = ma = m\frac{dv}{dt} = 0, \tag{2}$$

at terminal velocity (no further acceleration). Equation (2) then becomes a quadratic in v_t which is trivial to solve:

$$v_\mathrm{t} = \frac{-c_1 \pm \sqrt{c_1^2 + 4c_2 mg}}{2c_2} = \boxed{\sqrt{\frac{mg}{c_2} + \left(\frac{c_1}{2c_2}\right)^2} - \frac{c_1}{2c_2},}$$

as desired. Note that the positive root is chosen so that $v_\mathrm{t} > 0$ and thus points downward.

Problem 5 (FC 2.18, modified) The force acting on a particle of mass m is given by $F = \alpha\sqrt{v}\,t$, where $\alpha > 0$ is a constant. If m passes through the origin ($x = 0$) with speed v_0 at $t = 0$, find $x(t)$, the position of m as a function of time.

Solution: This is an application of problem 2.17 (F&C, ed. 7). We have:

$$F = \alpha\sqrt{v}\,t = ma = m\frac{dv}{dt},$$

which is a separable first order ODE. Thus,

$$\frac{dv}{v^{1/2}} = \frac{\alpha}{m} t\,dt \quad \Rightarrow \quad \int v^{-1/2} dv = \frac{\alpha}{m}\int t\,dt \quad \Rightarrow \quad 2\sqrt{v} = \frac{\alpha}{2m}t^2 + c, \tag{1}$$

where c is a constant of integration evaluated using *initial* conditions $v = v_0$ at $t = 0$. Thus, $c = 2\sqrt{v_0}$, and equation (1) becomes:

$$\sqrt{v} = \frac{\alpha}{4m}t^2 + \sqrt{v_0} \quad \Rightarrow \quad v(t) = \left(\frac{\alpha}{4m}t^2 + \sqrt{v_0}\right)^2 = \frac{\alpha^2}{16m^2}t^4 + \frac{\alpha}{2m}t^2\sqrt{v_0} + v_0.$$

But,

$$v(t) = \frac{dx}{dt} \quad \Rightarrow \quad dx = v(t)dt \quad \Rightarrow \quad \int dx = \int \left(\frac{\alpha^2}{16m^2}t^4 + \frac{\alpha}{2m}t^2\sqrt{v_0} + v_0\right)dt$$

$$\Rightarrow \quad x = \frac{\alpha^2}{80m^2}t^5 + \frac{\alpha}{6m}t^3\sqrt{v_0} + v_0 t + d,$$

where d is another constant of integration. Since $x = 0$ at $t = 0$, $d = 0$, and we have finally:

$$\boxed{x(t) = \frac{\alpha^2}{80m^2}t^5 + \frac{\alpha}{6m}t^3\sqrt{v_0} + v_0 t.}$$
