Solutions to Assignment 3

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1 Consider the following linear, second-order, homogeneous ODE:

$$y''(x) - 4y'(x) \;=\; 0. \tag{1}$$

a) Find "by inspection" two linearly independent solutions to equation (1).

b) From your two linearly independent solutions, write down the general solution.

c) Show that when the boundary conditions $y(0) = -1$ and $y'(0) = -2$ are applied to your general solution in part b, you get:

$$y(x) \;=\; -e^{2x}\cosh 2x,$$

where $\cosh z \equiv \frac{1}{2}(e^z + e^{-z})$ is the *hyperbolic cosine function*, which we'll meet in an upcoming class.

Solution: a) Equation (1) includes a first and second derivative of y but not y itself. Thus, the most obvious solution to equation (1) is a constant, $y = c$, for then $y' = y'' = 0$, and equation (1) is satisfied trivially. So what constant do we choose? It doesn't matter; Occam might suggest $y_1(x) = 1$, so let's go with that.

As for the second solution, it may be a bit more obvious if we let $z = y'$ and rewrite equation (1) as:

$$z' - 4z \;=\; 0 \quad\Rightarrow\quad z' \;=\; 4z,$$

and ask the question: *What function, $z(x)$, has a first derivative equal to itself times 4?* Answer: $z(x) = e^{4x}$, for then,

$$\frac{dz}{dx} \;=\; \frac{d}{dx}e^{4x} \;=\; 4e^{4x} \;=\; 4z.$$

But if $z(x) = e^{4x}$, then,

$$y(x) \;=\; \int z(x)\,dx \;=\; \frac{e^{4x}}{4},$$

and $\frac{1}{4}e^{4x}$ also solves equation (1). But if $\frac{1}{4}e^{4x}$ solves equation (1), so will e^{4x} (leading constants don't matter in a homogeneous ODE—try it!), and our two solutions are:

$$\boxed{y_1(x) \;=\; 1} \qquad \text{and} \qquad \boxed{y_2(x) \;=\; e^{4x}.} \tag{2}$$

Evidently, there is no constant, α, such that $y_1 = \alpha y_2$, and the two solutions in equation (2) are linearly independent.

b) The general solution to equation (1) is a linear combination of the two linearly independent solutions in equations (2). That is,

$$\boxed{y(x) \;=\; Ay_1(x) + By_2(x) \;=\; A + Be^{4x},} \tag{3}$$

where A and B are constants (independent of x).

c) Since one of the boundary conditions is applied to y', we first differentiate equation (3) to get:

$$y'(x) \;=\; 4Be^{4x}. \tag{4}$$

Then, setting $y(0) = -1$ and $y'(0) = -2$, we get:

$$y(0) \;=\; A + B \;=\; -1 \quad \text{and} \quad y'(0) \;=\; 4B \;=\; -2 \quad \Rightarrow \quad A \;=\; B \;=\; -\frac{1}{2},$$

which, when substituted into equation (3), gives:

$$y(x) \;=\; -\frac{1}{2}\big(1 + e^{4x}\big) \;=\; -e^{2x}\underbrace{\frac{e^{2x} + e^{-2x}}{2}}_{\cosh 2x} \quad \Rightarrow \quad \boxed{y(x) \;=\; -e^{2x}\cosh 2x,}$$

as desired.

Incidentally, equation (1) could also be solved by breaking it up into two first order, separable ODEs. Letting, as we did above, $z = y'$, equation (1) becomes the 1st order ODE:

$$\frac{dz}{dx} - 4z \;=\; 0 \quad \Rightarrow \quad \frac{dz}{z} \;=\; 4dx \quad \Rightarrow \quad \int \frac{dz}{z} \;=\; 4\int dx \quad \Rightarrow \quad \ln z \;=\; 4x + \beta,$$

where β is a constant of integration. Thus,

$$z(x) \;=\; e^{4x+\beta} \;=\; be^{4x},$$

where $b = e^{\beta}$. But $y' = z$, and we have the second separable 1st order ODE:

$$\frac{dy}{dx} \;=\; be^{4x} \quad \Rightarrow \quad dy \;=\; be^{4x}dx \quad \Rightarrow \quad \int dy \;=\; b\int e^{4x}dx \quad \Rightarrow \quad \boxed{y(x) \;=\; A + Be^{4x},} \tag{5}$$

where $B = b/4$ and A is a second constant of integration.

Equation (5) is identical to equation (3), and thus if you followed this path, you would have answered part b before part a. To answer part a, you would have had to glean the two independent solutions from equation (5), namely $y_1(x) = 1$ and $y_2(x) = e^{4x}$. Part c then follows directly from equation (5) as above.

Problem 2 (FC 3.2 & 3.3)

a) A piston executes simple harmonic motion with an amplitude of 0.100 m. If it passes through the centre of its motion with a speed $0.500\,\mathrm{m\,s^{-1}}$, find the period of oscillation.

b) A particle undergoes simple harmonic motion at a frequency of 20.0 Hz. Find the displacement at any time, t, for the initial conditions ($t = 0$) $x(0) = 0.100\,\mathrm{m}$ and $\dot{x}(0) = 0.250\,\mathrm{m\,s^{-1}}$.

Solution: a) For a piston executing simple harmonic motion, start with equation (2.2.5) in the class notes:

$$x(t) \;=\; x_0 \cos\omega_0 t \quad\Rightarrow\quad \dot{x}(t) \;=\; -x_0\omega_0 \sin\omega_0 t.$$

At the centre of its motion, the piston passes through its equilibrium point where its speed is maximum. Thus, $\sin\omega_0 t = 1$, $v_{\max} = x_0\omega_0$ (the negative sign just means the speed is in the opposite direction of the displacement), and we have:

$$\omega_0 \;=\; \frac{v_{\max}}{x_0} \;=\; \frac{0.500}{0.100} \;=\; 5.00\,\mathrm{rad\,s^{-1}},$$

using the values given in the problem. Therefore, the period of oscillations is:

$$T \;=\; \frac{2\pi}{\omega_0} \;=\; \frac{2\pi}{5.00} \;\sim\; \underline{\underline{1.26\,\mathrm{s}}}.$$

b) Starting this time from equation (2.2.3) in the class notes,

$$x(t) \;=\; A\cos\omega_0 t + B\sin\omega_0 t \quad\Rightarrow\quad \dot{x}(t) \;=\; -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t,$$

we apply boundary conditions to find A and B. In this case, at $t = 0$,

$$\begin{aligned} x(0) \;&=\; 0.100 \;=\; A\cancelto{1}{\cos(0)} + B\cancelto{0}{\sin(0)} \;=\; A; \\ \dot{x}(0) \;&=\; 0.250 \;=\; -A\omega_0\cancelto{0}{\sin(0)} + B\omega_0\cancelto{1}{\cos(0)} \;=\; B\omega_0 \quad\Rightarrow\quad B \;=\; \frac{0.250}{2\pi f} \;=\; \frac{6.25\times10^{-3}}{\pi}\,\mathrm{m}. \end{aligned}$$

Thus,

$$\boxed{x(t) \;=\; 0.100\cos(40\pi t) + \frac{6.25\times10^{-3}}{\pi}\sin(40\pi t),}$$

or,

$$x(t) \;=\; 0.100\cos(126.\,t) + 1.99\times10^{-3}\sin(126.\,t).$$

Alternately, one could start with,

$$x(t) \;=\; x_0\cos(\omega_0 t - \phi_0) \quad\Rightarrow\quad \dot{x}(t) \;=\; -\omega_0 x_0 \sin(\omega_0 t - \phi_0).$$

Then, at $t = 0$:

$$x(0) \;=\; x_0\cos(-\phi_0) \;=\; x_0\cos\phi_0 \;=\; \frac{1}{10}; \tag{1}$$

$$\dot{x}(0) \;=\; -\omega_0 x_0\sin(-\phi_0) \;=\; \omega_0 x_0\sin\phi_0 \;=\; \frac{1}{4} \quad\Rightarrow\quad x_0\sin\phi_0 \;=\; \frac{1}{4\omega_0}. \tag{2}$$

Dividing Eq. (2) by Eq. (1), we get:

$$\frac{\sin\phi_0}{\cos\phi_0} \;=\; \tan\phi_0 \;=\; \frac{10}{4\omega_0} \;=\; \frac{5}{4\pi f} \;=\; \frac{1}{16\pi} \quad\Rightarrow\quad \phi_0 \;\sim\; 0.0199\,\text{rad},$$

for $f = 20\,\text{Hz}$. Then, squaring both sides of Eq. (1) and (2) and adding, we get:

$$x_0^2\underbrace{(\cos^2\phi_0 + \sin^2\phi_0)}_{1} \;=\; \frac{1}{100} + \frac{1}{16\omega_0^2}$$

$$\Rightarrow\quad x_0 \;=\; \frac{\sqrt{16\omega_0^2 + 100}}{40\omega_0} \;=\; \frac{\sqrt{64\pi^2 f^2 + 100}}{80\pi f} \;=\; \frac{\sqrt{(160\pi)^2 + 100}}{1{,}600\pi} \;\sim\; 0.100$$

for $f = 20\,\text{Hz}$. Thus,

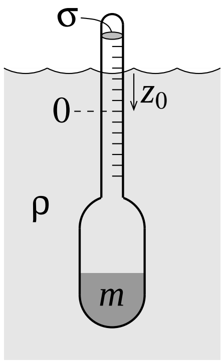
$$\boxed{x(t) \;\sim\; 0.100\cos(126.\,t - 0.0199),}$$

to three significant figures. Evidently, expressing $x(t)$ in this fashion is a little more awkward than the first.

Problem 3 This problem is based on *Archimedes' principle*, which states that the *buoyancy force* acting upward on an object floating or immersed in a liquid is equal to the weight of the displaced liquid.

An hydrometer is a device used to measure the density of a liquid based Archimedes' principle. It consists of a sealed, graduated hollow glass tube of uniform cross section, σ, and a ballast attached to the bottom for stability.

An hydrometer of mass m is lowered into a liquid of density ρ so that it floats freely. It is then depressed a distance z_0 and released.



a) Ignoring the viscosity of the liquid, show that the hydrometer oscillates as a simple harmonic oscillator with a period given by,

$$T \;=\; 2\pi\sqrt{\frac{m}{\rho\sigma g}}.$$

For the experimentalists, the liquid density can be calculated with:

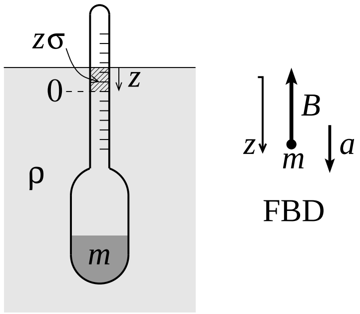
$$\rho \;=\; \frac{4\pi^2 m}{T^2\sigma g},$$

where T can be measured with a stopwatch.

b) Given the generic equation of motion for an SHO, $m\ddot{z} + kz = 0$, what is the effective spring constant, k, of the hydrometer?

Hint: Our discussion on vertical springs in §2.2 of the lecture notes shows that all forces establishing the equilibrium position of the system can be ignored in the FBD.

Solution: a) Following the hint, we ignore all forces establishing the equilibrium position of the hydrometer, and define the location on the hydrometer tube that intersects the liquid level at equilibrium to be zero. When the hydrometer sinks ($z > 0$) or rises ($z < 0$) an additional distance $-z_0 \le z \le z_0$ into the liquid, an additional volume $z\sigma$ of liquid is displaced (hatched area in inset), giving rise to an unbalanced buoyancy force, $B = \rho z \sigma g$, as indicated on the FBD. Thus,



$$-B = ma \quad \Rightarrow \quad -\rho z \sigma g = m\ddot{z} \quad \Rightarrow \quad \ddot{z} = -\frac{\rho \sigma g}{m}\, z,$$

which is the equation of motion for an SHO, with frequency,

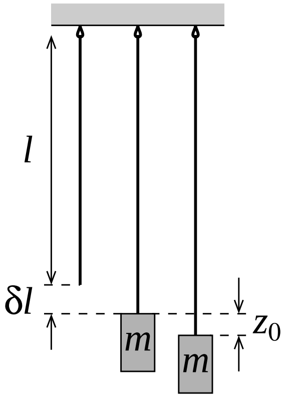
$$\omega_0^2 = \frac{\rho \sigma g}{m} \quad \Rightarrow \quad \boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{\rho\sigma g}},}$$

as desired.

b) For a Hooke's spring, $\omega_0^2 = k/m$. Thus, the effective spring constant, k, for the hydrometer is given by:

$$\boxed{k = \omega_0^2 m = \rho \sigma g.}$$

Problem 4 In this problem, you will learn about the elasticity of some materials, and an important example of simple harmonic motion not discussed in class. One might argue this falls under the realm of materials science or even engineering, but any good physicist should at least be aware of what is known as *Young's modulus*.



Consider a wire of length l, cross sectional area A, and negligible mass hanging vertically from a fixed anchor. If a mass m is hung from the free end, the wire stretches by an amount δl, as shown in the inset. Typically, $\delta l \ll l$ (*e.g.*, a thin metal wire will break before δl gets too big), although for a rubber band, δl could be comparable to and even greater than l.

Define the *strain*, Σ (the Greek capital 'S'), as the *distortion* of the wire caused by the weight of the hanging mass, m:

$$\Sigma \equiv \frac{\delta l}{l}, \tag{1}$$

a unitless quantity. Next, define the *stress* on the wire, S, as the *applied force per unit area* on the wire:

$$S \equiv \frac{F}{A}, \tag{2}$$

where $F = mg$ in this case. Thus, S has the units of *pressure*.

So far, everything has been definitions. Here's the only bit of physics: Experimentally, for *elastic* materials (which includes steel, by the way), *the stress is proportional to the strain*:

$$S \propto \Sigma \quad \Rightarrow \quad S = -Y\Sigma, \tag{3}$$

where the proportionality constant, $Y > 0$, is *Young's modulus* (units $\mathrm{N\,m^{-2}}$), a property of the material making up the wire (like its density or electrical conductivity). The negative sign means the stress and strain act in opposite directions. In the present example, the distortion is downward while the restoring force (the tension in the wire) acts upward.

Note that equation (3) is an approximation for how elastic materials actually behave, with the approximation better for smaller $\delta l/l$. Young's moduli for several common materials are given in the table below in units of GPa (1 gigaPascal = $10^9\,\mathrm{N\,m^{-2}}$).

rubber band	0.01	glass	70	copper	117
aluminum	69	brass	110	steel	200

a) If m is pulled down an additional distance $z_0 \ll l$ then released, show that in the absence of any dissipative (frictional) forces, m moves up and down as a simple harmonic oscillator with a period given by:

$$T = 2\pi\sqrt{\frac{ml}{YA}}. \tag{4}$$

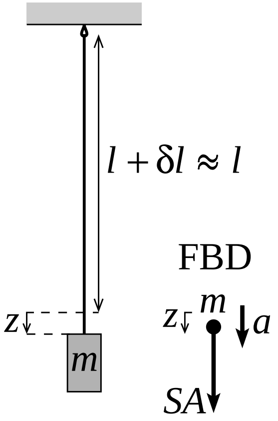
You may assume $\delta l \ll l$ and thus $l + \delta l \approx l$.

Hint: Review what we did in class for vertical oscillators (§2.2 in the class notes).

b) Find a numerical value for the period of oscillation for $m = 1.00\,\mathrm{kg}$ hanging on a length $l = 1.00\,\mathrm{m}$ of 12-gauge copper wire (2.05 mm in diameter). Why do you suppose this would be difficult to demonstrate in class?

c) By considering the equilibrium state of m hanging on the wire, show that the period given in equation (4) is the same as that of a simple pendulum of length δl.

Solution: a) From the discussion on vertical oscillators in §2.2 of the class notes, if we measure displacements from the equilibrium point of m hanging on the wire (length $l + \delta l \approx l$), then we need only consider the unbalanced force caused by the additional distortion $-z_0 \le z \le z_0$, as shown in the inset. Thus, from the FBD,



$$F = m\ddot{z} = SA = -Y\Sigma A = -Y\frac{z}{l+\delta l}A \approx -\frac{YA}{l}z,$$

using equation (2) to set $F = SA$, equation (3) to set $S = -Y\Sigma$, and equation (1) for Σ. Thus,

$$\boxed{\ddot{z} = -\frac{YA}{ml}z,}$$

the equation of motion for a simple harmonic oscillator. The frequency and period of oscillation are therefore given by:

$$\omega_0^2 = \frac{YA}{ml} \quad \Rightarrow \quad \boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{ml}{YA}},}$$

which is equation (4).

b) A 12-gauge wire has a diameter $d = 2.05\,\mathrm{mm}$, and thus a cross sectional area,

$$A = \pi\frac{d^2}{4} = \pi\frac{(2.05)^2}{4} \sim 3.301\,\mathrm{mm}^2 = 3.301\times10^{-6}\,\mathrm{m}^2,$$

retaining four significant figures for an intermediate result. Thus,

$$T = 2\pi\sqrt{\frac{(1.00)(1.00)}{(1.17\times10^{11})(3.301\times10^{-6})}} \sim 0.0101\,\mathrm{s},$$

or about 99 Hz (oscillations per second). It is for this reason that demonstrating such oscillations is difficult in a classroom setting.

c) When m is hanging from the spring in equilibrium, its weight is balanced by the restoring force of the stretched wire. Thus,

$$mg + SA = 0 \quad \Rightarrow \quad mg = -SA = Y\Sigma A = Y\frac{\delta l}{l}A \quad \Rightarrow \quad \frac{ml}{YA} = \frac{\delta l}{g}.$$

Substituting this into equation (4), we get:

$$\boxed{T = 2\pi\sqrt{\frac{\delta l}{g}},}$$

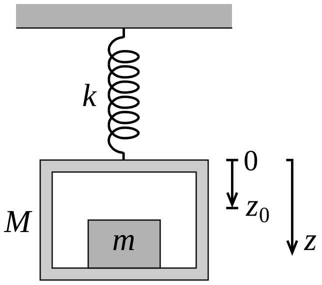
which is the period of a simple pendulum of length δl (equation 2.2.7 in the class notes).

Problem 5 (FC 3.8) A spring with spring constant k supports a box of mass M from the ceiling. Inside the box rests a block of mass m. The system is pulled down a distance z_0 from the equilibrium position and then released.

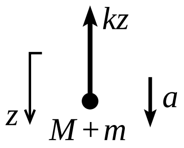
a) Find the normal force exerted by the bottom of the box on the block as a function of time.

Hint: Review the discussion on vertical springs in §2.2 of the class notes.

b) For what value of z_0 does the block just begin to leave the bottom of the box at the top of the oscillation?



Solution: a) Using the hint, we ignore the weight of the $M + m$ system and the stretch in the spring required to balance it. Thus, defining the equilibrium point as $z = 0$, the free body diagram FBD 1 for the $M + m$ system shows the single unbalanced force, kz, when the system is displaced by $-z_0 \leq z \leq z_0$ from equilibrium. Newton's 2nd then gives us:



FBD 1

$$-kz = (M+m)\ddot{z} \quad \Rightarrow \quad \ddot{z} = -\frac{k}{M+m}\,z, \tag{1}$$

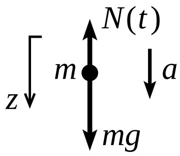
an equation for a simple harmonic oscillator whose solution is,

$$z(t) = z_0 \cos \omega_0 t, \tag{2}$$

(equation 2.2.5 in the class notes) where,

$$\omega_0 = \sqrt{\frac{k}{M+m}}, \tag{3}$$

is the oscillation frequency.



FBD 2

Now, consider the free body diagram for m alone (FBD 2). Here, there are two forces acting directly on m, namely mg and N, the normal force exerted by M on m and the quantity we seek. Thus, from Newton's 2nd law, we have:

$$-N + mg = m\ddot{z} \quad \Rightarrow \quad N = m(g - \ddot{z}). \tag{4}$$

Now, from equation (2), we have,

$$\dot{z} = -z_0\omega_0 \sin \omega_0 t \quad \Rightarrow \quad \ddot{z} = -z_0\omega_0^2 \cos \omega_0 t.$$

Substituting this into equation (4), we get,

$$\boxed{N(t) = m\big(g + z_0\omega_0^2 \cos \omega_0 t\big),} \tag{5}$$

where ω_0 is given by equation (3).

b) Equation (2) is a solution to equation (1) for initial conditions $z = z_0$ and $\dot{z} = 0$ at $t = 0$, where positive z points *downward*. Thus, at the bottom of the oscillation, $\omega_0 t = 0, 2\pi, 4\pi, \ldots$

while at the top of the oscillation, $\omega_0 t = \pi, 3\pi, 5\pi, \ldots$. Thus, taking $\omega_0 t = \pi \Rightarrow \cos\omega t = -1$, equation (5) becomes:

$$N_{\text{top}} \;=\; m(g - z_0\omega_0^2).$$

For m to just leave the bottom of the box at the top of the oscillation, $N_{\text{top}} = 0$ and,

$$g \;=\; z_0\omega_0^2 \quad \Rightarrow \quad \boxed{z_0 \;=\; \frac{g}{\omega_0^2} \;=\; \frac{(M+m)g}{k},}$$

using equation (3). This is the minimum amplitude of oscillation that will make m feel weightless at the top of oscillation.