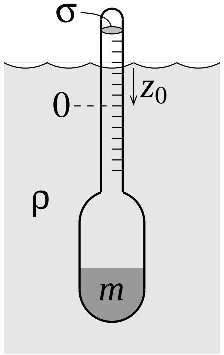
Solutions to Assignment 4

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1 Consider the hydrometer problem of the previous assignment.



A hydrometer of mass m and cross section σ is placed carefully into a liquid of density ρ, then set into oscillation with initial amplitude z_0. It is noted that after 12 complete oscillations, the amplitude of oscillation has fallen by a factor of 2.

From equation (2.4.1) in the class notes, the equation of motion for a damped harmonic oscillator is: $\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = 0$. Using the result of the previous assignment, namely,

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{\rho\sigma g}}, \qquad (1)$$

find the effective "damping coefficient", γ, for the oscillating hydrometer.

Solution: For an underdamped oscillator with initial amplitude z_0, the amplitude as a function of time is given by (equation 2.4.11 from the class notes),

$$A(t) = z_0 e^{-\gamma t}.$$

Thus, time for amplitude to fall by 1/2, $t_{1/2}$, is given by:

$$A(t_{1/2}) = \frac{z_0}{2} = z_0 e^{-\gamma t_{1/2}} \quad \Rightarrow \quad \gamma t_{1/2} = \ln 2 \quad \Rightarrow \quad t_{1/2} = \frac{\ln 2}{\gamma}.$$

Now, the number of complete periods for the amplitude to fall to $z_0/2$ is 12, and thus,

$$\frac{t_{1/2}}{T_0} = 12 = \frac{\ln 2}{\gamma}\frac{1}{2\pi}\sqrt{\frac{\rho\sigma g}{m}} \quad \Rightarrow \quad \boxed{\gamma = \frac{\ln 2}{24\pi}\sqrt{\frac{\rho\sigma g}{m}},}$$

using equation (1) for T_0.

Problem 2 Starting from the general solution for the damped harmonic oscillator derived in class [equation (2.4.2) in the class notes], namely,

$$x(t) = e^{-\gamma t}\left(Ae^{qt} + Be^{-qt}\right), \qquad (1)$$

where, as defined in class, $\gamma = b/2m$ is the damping coefficient, $q = \sqrt{\gamma^2 - \omega_0^2}$, and $\omega_0 = \sqrt{k/m}$ is the oscillator frequency when $b = 0$, use the initial conditions,

$$x(0) \;=\; 0 \qquad \text{and} \qquad \dot{x}(0) \;=\; v_0,$$

to show that,

$$x(t) \;=\; v_0 \begin{cases} e^{-\gamma t}\,\dfrac{\sinh qt}{q}, & \gamma > \omega_0 \text{ (overdamped);} \\ te^{-\omega_0 t}, & \gamma = \omega_0 \text{ (critically damped);} \\ e^{-\gamma t}\,\dfrac{\sin \omega_{\mathrm{d}} t}{\omega_{\mathrm{d}}}, & \gamma < \omega_0 \text{ (underdamped);} \\ \dfrac{\sin \omega_0 t}{\omega_0}, & \gamma = 0 \text{ (undamped).} \end{cases}$$

These initial conditions are tantamount to giving the oscillator a sharp blow (impulse) from the equilibrium point of the spring at $t = 0$, resulting in an initial velocity of v_0.

Solution: As was done in class, we differentiate equation (1) to get:

$$\dot{x}(t) \;=\; -\gamma e^{-\gamma t}\left(Ae^{qt} + Be^{-qt}\right) + e^{-\gamma t} q \left(Ae^{qt} - Be^{-qt}\right), \tag{2}$$

and apply the new initial conditions to equations (1) and (2) to find A and B. To that end,

$$x(0) \;=\; A + B \;=\; 0 \qquad \text{and} \qquad \dot{x}(0) \;=\; -\gamma\cancelto{0}{(A+B)} + q(A - B) \;=\; v_0$$

$$\Rightarrow \quad A \;=\; -B \;=\; \frac{v_0}{2q}$$

Thus, equation (1) becomes:

$$x(t) \;=\; e^{-\gamma t}\frac{v_0}{2q}(e^{qt} - e^{-qt}) \;=\; \frac{v_0}{q}\,e^{-\gamma t}\sinh qt \quad (q \neq 0), \tag{3}$$

using the definition of the hyperbolic sine given by equation (2.4.8) in the class notes.

As done in class, we consider four cases.

Case 1: For $q > 0 \in \mathbb{R}$, $\gamma > \omega_0$ and the system is *over-damped*. This is described by equation (3) as written.

Case 2: For $q = 0$, $\gamma = \omega_0$ and the system is *critically damped*. For this, we must examine equation (3) in the limit as $q \to 0$. Thus,

$$\lim_{q\to 0} x(t) \;=\; v_0 e^{-\omega_0 t} \lim_{q\to 0} \frac{\sinh qt}{q} \;=\; v_0 e^{-\omega_0 t} \underbrace{\lim_{q\to 0} \frac{t\cosh qt}{1}}_{\text{l'Hôpital}} \;=\; v_0 t e^{-\omega_0 t}, \tag{4}$$

since $\cosh 0 = 1$.

Case 3: For $\gamma < \omega_0$, $q \in \mathbb{I}$ and the system is *underdamped*. For this, we let,

$$\omega_\mathrm{d} \;=\; \sqrt{\omega_0^2 - \gamma^2} \;=\; q/i,$$

where $\omega_\mathrm{d} \in \mathbb{R}$ is the natural oscillation frequency of the underdamped oscillator. Thus, $q = i\omega_\mathrm{d}$ and equation (3) becomes:

$$x(t) \;=\; \frac{v_0}{i\omega_\mathrm{d}}\, e^{-\gamma t} \sinh i\omega_\mathrm{d} t \;=\; \frac{v_0}{\omega_\mathrm{d}}\, e^{-\gamma t} \sin \omega_\mathrm{d} t, \tag{5}$$

using one of equations (2.4.9) in the class notes.

Case 4: For $\gamma = 0$, $\omega_\mathrm{d} = \omega_0$ and the system is *undamped*, in which case,

$$\lim_{\gamma\to 0} x(t) \;=\; \frac{v_0}{\omega_0} \sin \omega_0 t. \qquad (6).$$

Bringing all four cases together, we have:

$$\boxed{x(t) \;=\; v_0 \begin{cases} e^{-\gamma t}\,\dfrac{\sinh qt}{q}, & \gamma > \omega_0 \text{ (overdamped);} \\ te^{-\omega_0 t}, & \gamma = \omega_0 \text{ (critically damped);} \\ e^{-\gamma t}\,\dfrac{\sin \omega_\mathrm{d} t}{\omega_\mathrm{d}}, & \gamma < \omega_0 \text{ (underdamped);} \\ \dfrac{\sin \omega_0 t}{\omega_0}, & \gamma = 0 \text{ (undamped),} \end{cases}}$$

as desired.

Problem 3 (FC 3.13) Suppose the amplitude of an oscillator drops off by a factor of e^{-1} of its initial value after n cycles.

a) Show that the ratio of the period of oscillation to the period of the same oscillator with no damping is given by:

$$\frac{T_\mathrm{d}}{T_0} \;=\; \sqrt{1 + \frac{1}{4\pi^2 n^2}}.$$

b) What is the quality factor, Q_d?

Hint: You may use the result of problem 3.9, namely that the ratio of successive maxima of a damped oscillator is $e^{-\gamma T_\mathrm{d}}$, where $T_\mathrm{d} = 2\pi/\omega_\mathrm{d}$ is the period of oscillation.

Solution: a) Problem 3.9 shows that the ratio of successive maxima of a damped oscillator is $e^{-\gamma T_\mathrm{d}}$, where $T_\mathrm{d} = 2\pi/\omega_\mathrm{d}$ is the period of oscillation. Here, we are given that after n oscillations, the amplitude of the oscillator dies down by a factor of e^{-1}. That is,

$$\left(e^{-\gamma T_\mathrm{d}}\right)^n = e^{-n\gamma T_\mathrm{d}} = e^{-1} \quad\Rightarrow\quad n\gamma T_\mathrm{d} = 1 \quad\Rightarrow\quad \gamma = \frac{\omega_\mathrm{d}}{2\pi n}. \tag{1}$$

Now, from the class notes,

$$\omega_\mathrm{d} = \sqrt{\omega_0^2 - \gamma^2} \quad\Rightarrow\quad \omega_0 = \sqrt{\omega_\mathrm{d}^2 + \gamma^2}$$

$$\Rightarrow\quad T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\omega_\mathrm{d}^2+\gamma^2}} = \frac{2\pi/\omega_\mathrm{d}}{\sqrt{1+\gamma^2/\omega_\mathrm{d}^2}} = \frac{T_\mathrm{d}}{\sqrt{1+1/(2\pi n)^2}},$$

using equation (1). Thus,

$$\boxed{\frac{T_\mathrm{d}}{T_0} = \sqrt{1+\frac{1}{4\pi^2 n^2}},}$$

as desired.

b) From equation (2.4.16) in the class notes,

$$Q_\mathrm{d} = \frac{\omega_d}{2\gamma} = \frac{\omega_\mathrm{d}}{2}\frac{2\pi n}{\omega_\mathrm{d}} \quad\Rightarrow\quad \boxed{Q_\mathrm{d} = \pi n,}$$

using equation (1) for γ.

Problem 4 According to electromagnetic theory, an accelerated electron radiates energy at the rate of,

$$\dot{E} = -\frac{e^2 a^2}{6\pi\epsilon_0 c^3}, \tag{1}$$

where e is the electron charge, $a = \ddot{x}$ is the electron's instantaneous acceleration, ϵ_0 is the permittivity of free space, and c is the speed of light.

a) If an electron oscillates along a straight line with an angular frequency ω_0 and an amplitude A, show that the energy it radiates in one cycle is given by:

$$\Delta E = -\frac{e^2 A^2 \omega_0^3}{6\epsilon_0 c^3}.$$

You may assume the electron to be weakly damped. Thus, $\omega_\mathrm{d} \sim \omega_0$ and its motion is adequately described by $x(t) = A\cos(\omega_0 t)$ during a single cycle.

b) Show that the quality factor, Q_d, for the oscillating electron is given by,

$$Q_\mathrm{d} = \frac{3 m_\mathrm{e}\epsilon_0 c^3}{e^2 f},$$

where m_e is the mass of the electron and $f = \omega_0/2\pi$ is the frequency in Hz.

c) If the oscillating electron is emitting visible light with wavelength $\lambda = 5{,}000\,\text{Å}$ ($1\ \text{Å} = 10^{-10}\,\text{m}$), find a numerical value for Q_d.

d) How many periods of oscillation does electron undergo by the time its energy has fallen to $1/e$ of its original value? Was the initial assumption of weak dampening justified?

Solution: a) Assuming $x(t) = A\cos\omega_0 t$ is valid over a single cycle,

$$a = \ddot{x} = -A\omega_0^2\cos\omega_0 t \quad\Rightarrow\quad \dot{E} = -\frac{e^2A^2\omega_0^4}{6\pi\epsilon_0 c^3}\cos^2\omega_0 t,$$

using equation (1). To find ΔE over a single period, we integrate $\dot{E}$ from t to $t+T_\text{d} \sim t+T_0$:

$$\Delta E = \int_t^{t+T_0}\dot{E}dt = -\frac{e^2A^2\omega_0^3}{6\pi\epsilon_0 c^3}\int_t^{t+T_0}\cos^2\omega_0 t\,(\omega_0 dt) = -\frac{e^2A^2\omega_0^3}{6\pi\epsilon_0 c^3}\int_{\omega_0 t}^{\omega_0 t+2\pi}\cos^2\theta\,d\theta,$$

where $\theta = \omega_0 t$. Since $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, we have:

$$\Delta E = -\frac{e^2A^2\omega_0^3}{12\pi\epsilon_0 c^3}\underbrace{\int_{\omega_0 t}^{\omega_0 t+2\pi}(1+\cos 2\theta)d\theta}_{2\pi} \quad\Rightarrow\quad \boxed{\Delta E = -\frac{e^2A^2\omega_0^3}{6\epsilon_0 c^3},}$$

as desired.

b) Assuming again that $x(t) = A\cos\omega_0 t$ is valid over a single cycle, the energy of the oscillating electron (assuming a simple harmonic oscillator) is,

$$\begin{aligned} E &= \tfrac{1}{2}m_\text{e}\dot{x}^2 + \tfrac{1}{2}m_\text{e}\omega_0^2 x^2 = \tfrac{1}{2}m_\text{e}A^2\omega_0^2\sin^2\omega_0 t + \tfrac{1}{2}m_\text{e}\omega_0^2A^2\cos^2\omega_0 t \\ &= \frac{m_\text{e}\omega_0^2A^2}{2}\underbrace{(\sin^2\omega_0 t + \cos^2\omega_0 t)}_{1} \end{aligned}$$

Combining this with equation (1), we get,

$$Q_\text{d} = 2\pi\frac{E}{|\Delta E|} = 2\pi\frac{m_\text{e}\omega_0^2A^2}{2}\frac{6\epsilon_0 c^3}{e^2A^2\omega_0^3} = \frac{6\pi m_\text{e}\epsilon_0 c^3}{e^2\omega_0} \quad\Rightarrow\quad \boxed{Q_\text{d} = \frac{3m_\text{e}\epsilon_0 c^3}{e^2 f},} \tag{2}$$

where $\omega_0 = 2\pi f$, as desired.

c) For visible light with wavelength λ, its frequency (Hz) is given by $f = c/\lambda$, and equation (2) becomes,

$$\begin{aligned} Q_\text{d} = \frac{3(m_\text{e}c^2)\epsilon_0\lambda}{e^2} &\sim \frac{3(8.1870\times10^{-14}\,\text{J})(8.8542\times10^{-12}\,\text{C}^2\,\text{J}^{-1}\,\text{m}^{-1})(5.00\times10^{-7}\,\text{m})}{(1.6022\times10^{-19}\,\text{C})^2} \\ &\sim \underline{\underline{4.24\times10^7}}\ \text{(unitless)}. \end{aligned}$$

d) For a weakly damped oscillator, the time for the total energy to fall off by a factor of e is its e-folding time, τ_E, given by equation (2.4.15) in the class notes. Thus, if $T_0 = 1/f$ is the period of oscillation, the number of complete cycles during time τ_E is,

$$N = \frac{\tau_E}{T_0} = \frac{Q_\mathrm{d}}{2\pi} \sim \underline{\underline{6.74 \times 10^6}},$$

using equation (2.4.16) in the class notes. As it takes millions of oscillations for the energy to drop off by a factor of e, the assumption of weak dampening made at the outset is justified.

Problem 5 Solve by "inspection", "direct integration", and/or "trial exponentials" the linear, second-order, inhomogeneous ODE:

$$y''(x) - y'(x) - 2y(x) = f(x), \tag{1}$$

for boundary conditions $y(0) = 0$ and $y'(0) = 2$, where:

a) $f(x) = 1$;

b) $f(x) = -2e^{-2x}$.

As discussed in §2.5 of the course notes, to solve such an *inhomogeneous* $[f(x) \neq 0]$ equation with boundary conditions, you must:

1. find two linearly independent solutions to the *homogeneous* equation [with $f(x) = 0$]; call these $y_1(x)$ and $y_2(x)$;
2. construct the general solution to the homogeneous equation,

$$y_\mathrm{h}(x) = Ay_1(x) + By_2(x),$$

   where A and B are constants;
3. by inspection or trial exponentials, find a *particular* solution, $y_\mathrm{p}(x)$ [*anything* that solves equation (1)];
4. write down the general solution to equation (1),

$$y(x) = y_\mathrm{h}(x) + y_\mathrm{p}(x);$$

5. and finally, apply boundary conditions to evaluate A and B.

Solution: For each part, a and b, we first solve the homogeneous version of equation (1),

$$y_\mathrm{h}''(x) - y_\mathrm{h}'(x) - 2y_\mathrm{h}(x) = 0, \tag{2}$$

by “trial exponentials”. Thus, substitute the trial solution e^{rx} into equation (2) to get:

$$r^2\cancel{e^{rx}} - r\cancel{e^{rx}} - 2\cancel{e^{rx}} = 0 \quad \Rightarrow \quad r^2 - r - 2 = (r+1)(r-2) = 0,$$

factoring the quadratic (or we could have used the quadratic formula). Thus, $r = -1$ or 2 and our two linearly independent solutions to equation (2) are:

$$y_1(x) = e^{-x} \qquad \text{and} \qquad y_2(x) = e^{2x},$$

from which we construct the general solution to (the homogeneous) equation (2):

$$y_{\mathrm{h}}(x) = Ae^{-x} + Be^{2x}, \tag{3}$$

where A and B are constants to be determined from boundary conditions.

a) For $f(x) = 1$, equation (1) becomes:

$$y''(x) - y'(x) - 2y(x) = 1, \tag{4}$$

for which we search for *any* solution by inspection. Since all coefficients are constant, try $y_{\mathrm{p}} =$ constant $\Rightarrow y''_{\mathrm{p}} = y'_{\mathrm{p}} = 0$, and equation (4) becomes:

$$-2y_{\mathrm{p}} = 1 \quad \Rightarrow \quad y_{\mathrm{p}} = -\frac{1}{2}.$$

The general solution to equation (1) is thus,

$$y(x) = y_{\mathrm{h}}(x) + y_{\mathrm{p}} = Ae^{-x} + Be^{2x} - \frac{1}{2} \quad \Rightarrow \quad y'(x) = -Ae^{-x} + 2Be^{2x}.$$

Applying the given boundary conditions, we get:

$$y(0) = A + B - \frac{1}{2} = 0 \qquad \text{and} \qquad y'(0) = -A + 2B = 2.$$

Solving for A and B, we get:

$$A = -\frac{1}{3}, \quad B = \frac{5}{6} \quad \Rightarrow \quad \boxed{y(x) = -\frac{1}{3}e^{-x} + \frac{5}{6}e^{2x} - \frac{1}{2},} \tag{5}$$

is the specific solution to equation (1) for the given boundary conditions and $f(x) = 1$.

b) For $f(x) = -2e^{-2x}$, equation (1) becomes:

$$y''(x) - y'(x) - 2y(x) = -2e^{-2x}, \tag{6}$$

for which we search for *any* solution by trial exponentials. Noting that if $y_{\mathrm{p}} \propto e^{-2x}$, all x-dependence cancels out, we are motivated to try $y_{\mathrm{p}}(x) = Ce^{-2x}$, in which case equation (6) becomes:

$$4C\cancel{e^{-2x}} + 2C\cancel{e^{-2x}} - 2C\cancel{e^{-2x}} = -2\cancel{e^{-2x}} \quad \Rightarrow \quad 4C = -2 \quad \Rightarrow \quad C = -\frac{1}{2}.$$

The general solution to equation (1) is thus,

$$y(x) \;=\; y_{\mathrm{h}}(x) + y_{\mathrm{p}}(x) \;=\; Ae^{-x} + Be^{2x} - \frac{1}{2}e^{-2x} \quad \Rightarrow \quad y'(x) \;=\; -Ae^{-x} + 2Be^{2x} + e^{-2x}.$$

Applying the given boundary conditions, we get:

$$y(0) \;=\; A + B - \frac{1}{2} \;=\; 0 \qquad \text{and} \qquad y'(0) \;=\; -A + 2B + 1 \;=\; 2.$$

Solving for A and B, we get:

$$A \;=\; 0, \quad B \;=\; \frac{1}{2} \quad \Rightarrow \quad \boxed{y(x) \;=\; \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x} \;=\; \sinh 2x,} \tag{7}$$

is the specific solution to equation (1) for the given boundary conditions and $f(x) = -2e^{-2x}$.

As an exercise, you should check to make sure that equations (5) and (7) do indeed solve equation (1) for the respective $f(x)$, and that in each case, $y(0) = 0$ and $y'(0) = 2$.
