Solutions to Assignment 5

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1 (FC 3.10 modified) A damped harmonic oscillator with $m = 5.0\,\mathrm{kg}$, $k = 180.\,\mathrm{N\,m^{-1}}$, and $b = 30.0\,\mathrm{kg\,s^{-1}}$ is subject to a driving force $F_0 \cos\omega t$, where $F_0 = 25.0\,\mathrm{N}$.

a) What value of ω results in steady-state oscillations with maximum amplitude (in the asymptotic limit)?

b) For the value of ω found in part a, what is the maximum amplitude and the corresponding phase shift?

Solution: a) From equation (2.6.12) in the class notes,

$$\omega_\mathrm{r} = \sqrt{\omega_0^2 - 2\gamma^2},$$

where $\omega_0^2 = \dfrac{k}{m}$ and $\gamma = \dfrac{b}{2m}$. Thus, from the numbers given,

$$\omega_0 = \sqrt{\frac{180.}{5.00}} = 6.00 \qquad \text{and} \qquad \gamma = \frac{30.0}{2(5.00)} = 3.00$$

$$\Rightarrow \quad \omega_\mathrm{r} = \sqrt{36.0 - 2(9.00)} = \sqrt{18} = \underline{\underline{3\sqrt{2} \sim 4.24\,\mathrm{rad\,s^{-1}}}}.$$

b) Start by noting that,

$$\omega_\mathrm{d} = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{36.0 - 9.00} = \sqrt{27} = 3\sqrt{3}.$$

Thus, from equations (2.6.13) and (2.6.15) in the class notes, we have,

$$A_\mathrm{max} = \frac{F_0}{b\,\omega_\mathrm{d}} = \frac{(25.0)}{(30.0)(3\sqrt{3})} = \underline{\underline{0.160\,\mathrm{m}}},$$

$$\tan\phi_\mathrm{r} = \frac{\omega_\mathrm{r}}{\gamma} = \frac{3\sqrt{2}}{3} = \sqrt{2} \quad \Rightarrow \quad \phi_\mathrm{r} \sim \underline{\underline{0.955\,\mathrm{rad}}}.$$

Problem 2 (FC 3.18) Find the particular solution to the differential equation of motion for the driven, damped harmonic oscillator,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{-\alpha t} \cos\omega t, \tag{1}$$

and show that the phase and amplitude of the steady-state oscillations are given by:

$$\tan\phi = \frac{2\omega(\gamma-\alpha)}{\alpha^2-\omega^2-2\gamma\alpha+\omega_0^2};$$

$$A = \frac{F_0/m}{\sqrt{\left(\alpha^2-\omega^2-2\gamma\alpha+\omega_0^2\right)^2+4\omega^2(\gamma-\alpha)^2}}.$$

Hint: This is the same driving force used in class (equation 2.6.1 in the class notes), except for the factor $e^{-\alpha t}$. So how to proceed?

Well, from Euler's formula (equation 2.4.5 in the class notes),

$$e^{i\omega t} = \cos\omega t + i\sin\omega t,$$

and thus $\cos\omega t$ is the *real part* of $e^{i\omega t}$ [written as $\cos\omega t = \Re(e^{i\omega t})$] and $\sin\omega t$ is the *imaginary part* of $e^{i\omega t}$ [written as $\sin\omega t = \Im(e^{i\omega t})$]. Thus, $e^{-\alpha t}\cos\omega t = \Re(e^{(-\alpha+i\omega)t}) = \Re(e^{\beta t})$, where $\beta \equiv -\alpha + i\omega \in \mathbb{C}$, Therefore, for the particular solution, try assuming the form,

$$x_\mathrm{p}(t) = Ae^{\beta t - i\phi}, \tag{2}$$

rather than $A\cos(\omega t-\phi)$ as we did in class (equation 2.6.3), with the understanding that your actual solution will just be the real part of this.

So with that lead, plug equation (2) into equation (1), and see what happens. . .

Solution: Using the hint, we assume a solution of the form given in equation (2), and substitute it into the differential equation (1) to get:

$$\beta^2 A e^{\cancel{\beta t}-i\phi} + 2\gamma\beta A e^{\cancel{\beta t}-i\phi} + \omega_0^2 A e^{\cancel{\beta t}-i\phi} = \frac{F_0}{m}\cancel{e^{\beta t}}$$

$$\Rightarrow \quad \beta^2 + 2\gamma\beta + \omega_0^2 = \frac{F_0}{Am}e^{i\phi}. \tag{3}$$

Note that by making this educated guess for x_p, the time dependence has cancelled out, leaving us with a complex equation (real and imaginary parts) which we use to find A and ϕ as functions of ω.

In class with the driving force $F_0\cos\omega t$, we arrived at an equation of the form,

$$f(\omega)\cos\omega t = g(\omega)\sin\omega t,$$

and argued for this to be true, $f(\omega)$ and $g(\omega)$ each had to be zero (see §2.6 in the class notes). This gave us two equations which we used to find $A(\omega)$ and $\phi(\omega)$.

Here, equation (3) is complex, and for it to be true the real and imaginary parts must be separately equal. This will give us the two equations we need to find $A(\omega)$ and $\phi(\omega)$. To that end, we need to expose the real and imaginary parts of equation (3) explicitly and for that, we substitute $\beta = -\alpha + i\omega$ to get:

$$\begin{aligned}
(-\alpha + i\omega)^2 + 2\gamma(-\alpha + i\omega) + \omega_0^2 &= \frac{F_0}{Am}(\cos\phi + i\sin\phi) \\
\Rightarrow \quad (\alpha^2 - \omega^2 - 2\gamma\alpha + \omega_0^2) + i(-2\alpha\omega + 2\gamma\omega) &= \frac{F_0}{Am}\cos\phi + i\frac{F_0}{Am}\sin\phi,
\end{aligned}$$

where the real and imaginary parts are now fully exposed. Equating these, we get:

$$\alpha^2 - \omega^2 - 2\gamma\alpha + \omega_0^2 = \frac{F_0}{Am}\cos\phi \quad \text{(real parts)} \tag{4}$$

$$-2\alpha\omega + 2\gamma\omega = \frac{F_0}{Am}\sin\phi \quad \text{(imaginary parts).} \tag{5}$$

The most efficient way to solve equations such as these is to exploit trig identities. So first, divide equation (5) by equation (4) to get (note how this eliminates A and leaves just the one unknown ϕ):

$$\boxed{\tan\phi = \frac{2\omega(\gamma - \alpha)}{\alpha^2 - \omega^2 - 2\gamma\alpha + \omega_0^2},} \tag{6}$$

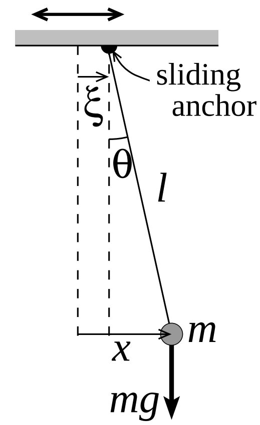
as desired. Next, square each of equations (4) and (5), and add them (to exploit $\cos^2\phi + \sin^2\phi = 1$ thereby eliminating ϕ and leaving an expression for A):

$$\begin{aligned}
\left(\alpha^2 - \omega^2 - 2\gamma\alpha + \omega_0^2\right)^2 + 4\omega^2(\gamma - \alpha)^2 &= \left(\frac{F_0}{Am}\right)^2 \underbrace{(\cos^2\phi + \sin^2\phi)}_{1} \\
\Rightarrow \quad \boxed{A(\omega) = \frac{F_0/m}{\sqrt{\left(\alpha^2 - \omega^2 - 2\gamma\alpha + \omega_0^2\right)^2 + 4\omega^2(\gamma - \alpha)^2}},} & \qquad (7)
\end{aligned}$$

as desired. Note how equations (6) and (7) reduce to equations (2.6.4) and (2.6.7) in the class notes when $\alpha \to 0$.

Problem 3 A simple pendulum of length l and bob mass m swings back and forth with an effective weak damping coefficient, γ. The anchor can be driven to slide back and forth horizontally in simple harmonic motion with a maximum amplitude ξ_0.

Parts a, b, and c were done in the tutorial. You may use these results to answer the next three parts.



d) Find numerical values for A/ξ_0 and ϕ when the system is driven at:

$$i)\ \ \omega = \frac{\omega_0}{2}; \qquad\qquad ii)\ \ \omega = 2\omega_0.$$

Note that you don't need to know g or l and thus ω_0!

For phases, choose values for ϕ such that $0 < \phi < \pi$ rad. For what driving frequency, ω, is the displacement of the bob in phase and out of phase with the driver?

e) Write down expressions for the resonant frequency, ω_r, and the amplitude at resonance, A_max, and from these, find numerical values to three significant figures for $1 - \omega_\mathrm{r}/\omega_0$ and A_max/ξ_0. Again, you don't need to know ω_0 for either of these values!

If the small angle approximation is to remain valid even at resonance (and thus $A_\mathrm{max} < 0.1l$, say), what is the maximum permitted value of ξ_0/l?

f) What is the phase, ϕ_r, at resonance? Now how would you say the phase of the bob relates to the driver?

Solution: d) *i*) For $\omega = \frac{1}{2}\omega_0$, we have from equations (3):

$$\frac{A}{\xi_0} = \frac{\cancel{\omega_0^2}}{\sqrt{\left(\frac{3}{4}\cancel{\omega_0^2}\right)^2 + \cancel{\omega_0^2}\left(\cancel{\omega_0}/100\pi\right)^2}} = \frac{1}{\sqrt{9/16 + 1/(10{,}000\pi^2)}} \sim \underline{\underline{\frac{4}{3} \sim 1.33.}}$$

$$\tan\phi = \frac{\cancel{\omega_0}(\cancel{\omega_0}/100\pi)}{\frac{3}{4}\cancel{\omega_0^2}} = \frac{4}{300\pi} \quad\Rightarrow\quad \phi = \tan^{-1}\left(\frac{4}{300\pi}\right) \sim \underline{\underline{0.00424\,\mathrm{rad}}}.$$

Here, $\phi \sim 0$ and the bob displacement is roughly in phase with the driver.

ii) For $\omega = 2\omega_0$, we have from equations (3):

$$\frac{A}{\xi_0} = \frac{\cancel{\omega_0^2}}{\sqrt{\left(-3\cancel{\omega_0^2}\right)^2 + 16\cancel{\omega_0^2}\left(\cancel{\omega_0}/100\pi\right)^2}} = \frac{1}{\sqrt{9 + 16/(10{,}000\pi^2)}} \sim \underline{\underline{\frac{1}{3} \sim 0.333.}}$$

$$\tan\phi = \frac{4\cancel{\omega_0}(\cancel{\omega_0}/100\pi)}{-3\cancel{\omega_0^2}} = -\frac{4}{300\pi} \quad\Rightarrow\quad \phi = \tan^{-1}\left(-\frac{4}{300\pi}\right) \sim \underline{\underline{3.137\,\mathrm{rad}}}.$$

Here, $\phi \sim \pi$ rad, and the bob displacement is nearly exactly out of phase with the driver.

e) With F_0/m replaced by $g\xi_0/l = \omega_0^2\xi_0$, we have from equations (2.6.12) and (2.6.13) in the class notes:

$$\boxed{\omega_\mathrm{r} = \sqrt{\omega_0^2 - 2\gamma^2}} \qquad \text{and} \qquad \boxed{A_\mathrm{max}(\gamma) = \frac{\omega_0^2\xi_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}}. \tag{4}$$

Thus, with $\gamma = \omega_0/(100\pi)$,

$$\omega_{\rm r} = \sqrt{\omega_0^2 - 2\frac{\omega_0^2}{(100\pi)^2}} = \omega_0\left(1 - \frac{2}{(100\pi)^2}\right)^{1/2} = \omega_0\left(1 - \frac{1}{2}\frac{2}{(100\pi)^2} + \cdots\right)$$

$$\Rightarrow \quad \frac{\omega_{\rm r}}{\omega_0} = 1 - \frac{1}{(100\pi)^2} + \cdots \quad \Rightarrow \quad 1 - \frac{\omega_{\rm r}}{\omega_0} \approx \frac{1}{(100\pi)^2} \sim \underline{\underline{1.013 \times 10^{-5}}}.$$

The resonant frequency is just one part in 10^5 smaller than the natural oscillation frequency of the pendulum.

Next, from the second of equations (4):

$$\frac{A_{\rm max}}{\xi_0} = \frac{\cancel{\omega_0^2}}{2\cancel{\omega_0}/100\pi}\frac{1}{\sqrt{\cancel{\omega_0^2} - \cancel{\omega_0^2}/(100\pi)^2}} = 50\pi\left(1 - \frac{1}{(100\pi)^2}\right)^{-1/2}$$

$$= 50\pi\left(1 + \frac{1}{2}\frac{1}{(100\pi)^2} + \cdots\right) = \underline{\underline{50\pi}} \text{ to within 5 parts in } 10^6.$$

Then, for $A_{\rm max} < 0.1l$,

$$A_{\rm max} = 50\pi\xi_0 < 0.1l \quad \Rightarrow \quad \frac{\xi_0}{l} < \frac{0.1}{50\pi} \sim \underline{\underline{6.37 \times 10^{-4}}}.$$

Thus, if the pendulum length is 1 m, ξ_0 can be no more than $\sim$0.64 mm for the "small angle" approximation to remain valid at resonant amplitude.

f) The resonant phase angle is given by substituting the first of equations (4) into the first of equations (3):

$$\tan\phi_{\rm r} = \frac{2\omega_{\rm r}\gamma}{\omega_0^2 - \omega_{\rm r}^2} = \frac{\cancel{2\gamma}\sqrt{\omega_0 - 2\gamma^2}}{\cancel{2}\gamma^{\cancel{2}}} = \sqrt{\omega_0^2 - 2\frac{\omega_0^2}{(100\pi)^2}}\,\frac{100\pi}{\omega_0} = \sqrt{(100\pi)^2 - 2}$$

$$= 100\pi \text{ to within one part in } 10^5$$

$$\Rightarrow \quad \phi_{\rm r} = \tan^{-1}(100\pi) \sim \underline{\underline{1.568\,\text{rad}}} = \frac{\pi}{2} \text{ to within two parts in } 10^3.$$

Thus, as expected for very weakly damped systems, the displacement of the bob is $\sim\pi/2$ radians out of phase with the driver, and thus half way between being in and out of phase.

Problem 4 (FC 3.11) A mass m moves along the x-axis subject to a restoring force $F_{\rm r} = -\frac{17}{2}\beta^2 mx$ and a drag force $F_{\rm d} = -3\beta m\dot{x}$, where x is the distance from the origin and β is a constant. In addition, a driving force, $F = mA\cos\omega t$ is applied to the particle along the x-axis, where A is another constant.

a) What value of ω results in steady-state oscillations about the origin ($x = 0$) with maximum amplitude?

b) For the value of ω found in part a, what is the maximum amplitude?

Solution: For the problem described, Newton's second law gives:

$$\sum F = F_\mathrm{r} + F_\mathrm{d} + F = -\frac{17}{2}\beta^2 m x - 3\beta m\dot{x} + mA\cos\omega t = m\ddot{x}$$

$$\Rightarrow \quad \ddot{x} + 3\beta\dot{x} + \frac{17}{2}\beta^2 x = A\cos\omega t,$$

is the differential equation of motion for the driven, damped oscillator. Comparing this to the generic ODE (equation 2.6.1 from the class notes),

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t,$$

we must have,

$$\gamma = \frac{3}{2}\beta, \qquad \omega_0^2 = \frac{17}{2}\beta^2, \qquad \text{and} \qquad F_0 = mA.$$

a) Here, we're looking for the resonant frequency given by equation (2.6.12) in the class notes,

$$\omega_\mathrm{r}^2 = \omega_0^2 - 2\gamma^2 = \frac{17}{2}\beta^2 - 2\left(\frac{3}{2}\beta\right)^2 = 4\beta^2 \quad \Rightarrow \quad \boxed{\omega_\mathrm{r} = 2\beta.}$$

b) Start by noting that,

$$\omega_\mathrm{d}^2 = \omega_0^2 - \gamma^2 = \frac{17}{2}\beta^2 - \left(\frac{3}{2}\beta\right)^2 = \frac{25}{4}\beta^2 \quad \Rightarrow \quad \omega_\mathrm{d} = \frac{5}{2}\beta.$$

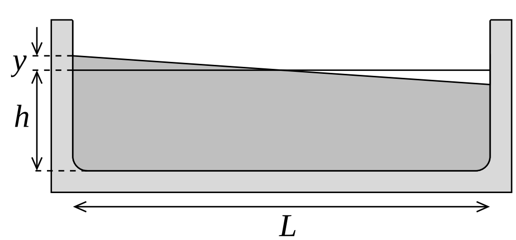
Thus, the resonant amplitude is given by equation (2.6.13) from the class notes:

$$A_\mathrm{max} = \frac{F_0}{b\,\omega_\mathrm{d}} = \frac{F_0}{2m\gamma\omega_\mathrm{d}} = \frac{\cancel{m}A}{\cancel{2}\cancel{m}\frac{3}{\cancel{2}}\beta\,\frac{5}{2}\beta} \quad \Rightarrow \quad \boxed{A_\mathrm{max} = \frac{2A}{15\beta^2}.}$$

Problem 5 As depicted in the figure below, a bathtub of length L is filled with water to a depth h.

a) If the water is driven to slosh back and forth so that the water surface remains planar, show that the potential energy of the bathwater, U, is given by:

$$U(y) = \frac{\rho L w g}{6}y^2,$$



where ρ is the density of water, w is the width of the tub, and y is the additional depth of water above h at the left end of the tub ($x = 0$).

b) If you choose to do the bonus part d below, you'll find that the kinetic energy, K, of the horizontal motion of the water is given by,

$$K(\dot{y}) = \frac{\rho w L^3}{60h}\dot{y}^2.$$

Assuming the oscillation is weakly damped, set $E = U + K =$ constant, and show that the water level rises and falls as a simple harmonic oscillator. Thus, find its natural frequency, ω_0, and period of oscillation, T_0.

For small damping, $\omega_\mathrm{r} \sim \omega_\mathrm{d} \sim \omega_0$. Therefore, what is the resonant period of oscillation for a tub with $L = 1.5\,\mathrm{m}$ and $h = 0.3\,\mathrm{m}$? Does this jibe with your intuition? (Surely *every* kid discovered that sliding back and forth along the bottom of the tub with just the "right" frequency will spill water onto the floor!)

c) The Bay of Fundy is about 250 km long. If we model it as a big bathtub sloshing back and forth as it is driven by solar and lunar tidal forces, the bay itself would represent half of the tub with the other half stuck in the Gulf of Maine. Thus, take $L \sim 500\,\mathrm{km}$ and use $h \sim 50\,\mathrm{m}$ as its effective depth. What is the resonant period of oscillation for the Bay of Fundy? Compare this with the driving frequency of the moon (two high tides per 24.9 hr, the time for the moon to return to the same spot in the sky), and in one sentence, explain why the Fundy tides are so notoriously high.

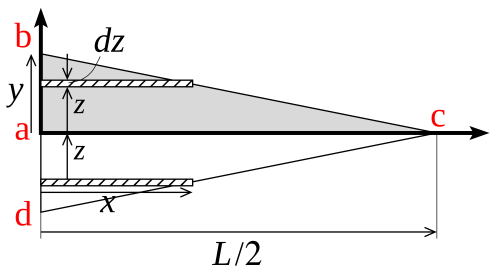
d) (5 bonus points) Show that the kinetic energy, K, of the horizontal motion of the water is given by:

$$K(\dot{y}) = \frac{\rho w L^3}{60h}\dot{y}^2.$$

Here, we assume the amplitude of oscillation is sufficiently small that only the back-and-forth motion of the water contributes to K, with the vertical motion contributing a negligible amount. And yet note that K as given is as a function of $\dot{y}$. Hmmm...

Hint: While part a should be a 3 or 4 liner, calculating K is a bit trickier and took me a full page to do so. The way I approached it was to calculate $v(x)$ (horizontal motion of the water) in terms of x, the horizontal position along the tub where $0 \le x \le L$, and $\dot{y}$, the rate at which the water level rises and falls at the left end of the tub. For this, I had to invoke the idea that the volume of water entering a fixed rectangle at location x of height h and width dx is the same as that leaving. Doing this, you should find $dv/dx = -\dot{z}/h$, where $\dot{z}$ is the speed at which water is rising or falling at point x. I then used a "similar triangles" argument to relate z and y. This is a tricky problem, so don't feel badly if you don't get it!

Solution: a) The potential energy of the water with the water level inclined is the same as though the mass in the triangle acd in the inset were raised to triangle abc. To do this, each horizontal strip of water of mass dm, length x, and thickness dz is raised a distance $2z$, where $0 \le z \le y$. Thus,



$$dU = 2gz\,dm \quad \Rightarrow \quad U = 2\rho wg \int_0^y xz\,dz,$$

since $dm = \rho xw\,dz$. Then, from similar triangles,

$$\frac{x}{y-z} = \frac{L/2}{y} \quad \Rightarrow \quad x = \frac{L(y-z)}{2y},$$

and we have:

$$U = \frac{\rho Lwg}{y}\int_0^y z(y-z)dz = \frac{\rho Lwg}{y}\left(\frac{yz^2}{2} - \frac{z^3}{3}\right)\Bigg|_0^y \quad \Rightarrow \quad \boxed{U(y) = \frac{\rho Lwg}{6}y^2,}$$

as desired. The fact that $U \propto y^2$ already tells us the system is a simple harmonic oscillator.

b) Given the result from part d, the total mechanical energy is:

$$E = U + K = \frac{\rho Lwg}{6}y^2 + \frac{\rho wL^3}{60h}\dot{y}^2,$$

which has the classical form for the mechanical energy of a simple harmonic oscillator. Since ω_0^2 is given by the ratio of the coefficients, we have:

$$\omega_0^2 = \frac{\rho Lwg}{6}\frac{60h}{\rho wL^3} = \frac{10gh}{L^2}$$

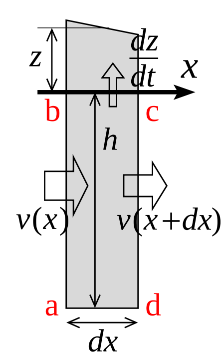
$$\Rightarrow \quad \boxed{\omega_0 = \frac{\sqrt{10gh}}{L}} \quad \text{and} \quad \boxed{T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi L}{\sqrt{10gh}}.}$$

For $L = 1.5\,\text{m}$ and $h = 0.3\,\text{m}$, $T_0 \sim 1.7\,\text{s}$, which seems about right given my distant memories of flooding the bathroom floor much to my mother's great consternation!

c) As for the mother of all bath tubs, the natural oscillation period of the Bay of Fundy, with $L = 5.0 \times 10^5\,\text{m}$ and an effective depth of $h = 50\,\text{m}$, is $T_0 \sim 4.49 \times 10^4\,\text{s} \sim 12.47\,\text{hr}$, which is very close to half the time it takes for the moon to return to the same position in the sky ($24.9\,\text{hr}$).

Thus, the high tides in the Bay of Fundy can be explained as a resonance phenomenon.

d) (Bonus 5 points) For the kinetic energy, consider the inset, which depicts a narrow column of fluid of width dx and depth h below the level line (x-axis) and a trapezoid of average height z above the level line representing the rest of the water column up to the surface. As shown, the horizontal water speed at the left is given by $v(x)$, at the right $v(x+dx)$, and at the level surface, dz/dt, the rate at which the surface is either rising ($dz/dt > 0$) or falling ($dz/dt < 0$) at that place and moment.



In the fixed rectangle abcd of height h and width dx, as much water enters abcd in a given time, dt, as leaves (large arrows), and we write:

$$\cancel{\rho w} hv(x)\,\cancel{dt} - \cancel{\rho w} hv(x+dx)\,\cancel{dt} - \cancel{\rho w}\, dx\,\frac{dz}{dt}\,\cancel{dt} \;=\; 0$$

$$\Rightarrow \quad -\frac{v(x+dx)-v(x)}{dx} \;=\; \frac{1}{h}\frac{dz}{dt} \quad \Rightarrow \quad \frac{dv}{dx} \;=\; -\frac{1}{h}\dot{z}. \tag{1}$$

Again from similar triangles,

$$\frac{y-z}{x} \;=\; \frac{2y}{L} \quad \Rightarrow \quad z \;=\; y\left(1-\frac{2x}{L}\right) \quad \Rightarrow \quad \dot{z} \;=\; \dot{y}\left(1-\frac{2x}{L}\right),$$

since here, x is a fixed location and treated as a constant in the differentiation. Substituting this into equation (1), we get:

$$\frac{dv}{dx} \;=\; -\frac{\dot{y}}{h}\left(1-\frac{2x}{L}\right)$$

$$\Rightarrow \quad v(x) \;=\; \int \frac{dv}{dx}\,dx \;=\; -\frac{\dot{y}}{h}\int\left(1-\frac{2x}{L}\right)dx \;=\; -\frac{\dot{y}}{h}\left(x-\frac{x^2}{L}+c\right),$$

where c is a constant of integration. This is evaluated by imposing the boundary condition that at $x=0$ (left end of the tub), the horizontal water speed is zero:

$$v(0) \;=\; -\frac{\dot{y}}{h}c \;=\; 0 \quad \Rightarrow \quad c \;=\; 0.$$

Thus,

$$v(x) \;=\; \frac{\dot{y}}{h}\left(\frac{x^2}{L}-x\right). \tag{2}$$

Note that this also satisfies the right boundary condition, namely $v(L)=0$.

We are finally in a position to evaluate the kinetic energy:

$$dK \;=\; \frac{1}{2}v(x)^2 dm \;=\; \frac{\rho w h\,dx}{2}v(x)^2, \tag{3}$$

where we have ignored the portion of the trapezoid above the level line, as this is compensated by an equal amount of missing mass at the corresponding point in the right half of the tub. Thus, substituting equation (2) into (3), we find:

$$K \;=\; \int_0^L dK \;=\; \frac{\rho w h}{2}\left(\frac{\dot{y}}{h}\right)^2\int_0^L\left(\frac{x^2}{L}-x\right)^2 dx$$

$$= \frac{\rho w}{2h}\dot{y}^2\left(\frac{x^5}{5L^2}-\frac{x^4}{2L}+\frac{x^3}{3}\right)\Bigg|_0^L \;=\; \frac{\rho w}{2h}\dot{y}^2\frac{L^3}{30}$$

$$\Rightarrow \quad \boxed{K(\dot{y}) \;=\; \frac{\rho w L^3}{60h}\dot{y}^2,}$$

as desired.