

Solutions to Assignment 6

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1 (FC 4.2) By taking their curls, determine which of the following forces are conservative.

a) $\vec{F} = (x, y, z)$

b) $\vec{F} = (y, -x, z^3)$

c) $\vec{F} = (y, x, z^3)$

d) $\vec{F} = -kr^{-n}\hat{e}_r$, where \hat{e}_r is the unit radial vector in spherical polar coordinates.

Solution: In Cartesian coordinates, the curl is given by (Appendix F):

$$\nabla \times \vec{F} = (\partial_y F_z - \partial_z F_y)\hat{i} + (\partial_z F_x - \partial_x F_z)\hat{j} + (\partial_x F_y - \partial_y F_x)\hat{k}. \quad (1)$$

In spherical polar coordinates, the curl is given by:

$$\begin{aligned} \nabla \times \vec{F} = & \frac{\hat{e}_r}{r \sin \theta} (\partial_\theta (\sin \theta F_\phi) - \partial_\phi (F_\theta)) + \frac{\hat{e}_\theta}{r \sin \theta} (\partial_\phi F_r - \sin \theta \partial_r (r F_\phi)) \\ & + \frac{\hat{e}_\phi}{r} (\partial_r (r F_\theta) - \partial_\theta F_r), \end{aligned} \quad (2)$$

where \hat{e}_r , \hat{e}_θ , and \hat{e}_ϕ are unit vectors in the r , θ , and ϕ directions.

a) Thus, for $\vec{F} = (x, y, z)$, use equation (1) to get $\nabla \times \vec{F} = \vec{0}$, since all derivatives in the curl are “cross derivatives” (e.g., $\partial_i F_j$, $i \neq j$), and thus 0.

$$\Rightarrow \boxed{\vec{F} \text{ is conservative.}}$$

b) For $\vec{F} = (y, -x, z^3)$, $\nabla \times \vec{F} = (0, 0, -1-1) = -2\hat{k} \neq \vec{0}$.

$$\Rightarrow \boxed{\vec{F} \text{ is not conservative.}}$$

c) For $\vec{F} = (y, x, z^3)$, $\nabla \times \vec{F} = (0, 0, 1-1) = \vec{0}$.

$$\Rightarrow \boxed{\vec{F} \text{ is conservative.}}$$

d) For $\vec{F} = -kr^{-n}\hat{e}_r$ in spherical polar coordinates. Thus, using equation (2), we get:

$$\nabla \times \vec{F} = -\frac{\hat{e}_\theta}{r \sin \theta} \partial_\phi(kr^{-n}) + \frac{\hat{e}_\phi}{r} \partial_\theta(kr^{-n}) = 0 \quad \Rightarrow \quad \boxed{\vec{F} \text{ is conservative.}}$$

Note that we could have argued, as we did for part a), that since $F(r)$ is a function of r only, and since the curl, no matter the coordinate system, involves cross derivatives only, that the curl must be zero.

Problem 2 (FC 4.5 modified) Let C_1 be the path on the x - y plane that joins $(0, 0)$ and $(1, 1)$ directly by the parabolic path $y = \sqrt{x}$, and let C_2 be the path that joins $(0, 0)$ to $(0, 1)$ by the line $x = 0$, and then $(0, 1)$ to $(1, 1)$ by the line $y = 1$.

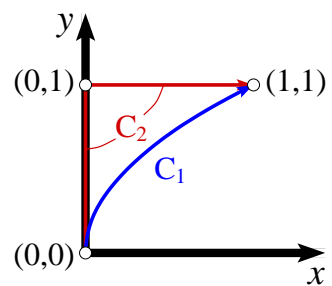
- a) Evaluate $\int \vec{F} \cdot d\vec{r}$ for $\vec{F} = xy\hat{i} + \frac{1}{2}x^2\hat{j}$ for each of the paths C_1 and C_2 , and verify that \vec{F} is conservative.
- b) Evaluate $\int \vec{F} \cdot d\vec{r}$ for $\vec{F} = xy\hat{i} + x^2\hat{j}$ for each of the paths C_1 and C_2 , and show that \vec{F} is not conservative.

Solution: For $\vec{F} = F_x\hat{i} + F_y\hat{j}$ and $d\vec{r} = dx\hat{i} + dy\hat{j}$,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (F_x dx + F_y dy). \quad (1)$$

a) Consider first $\vec{F} = xy\hat{i} + \frac{1}{2}x^2\hat{j}$. For path C_1 , $y = x^{1/2}$, $dy = \frac{1}{2}x^{-1/2}dx$, and equation (1) becomes:

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 (x^{3/2} dx + \frac{1}{4}x^{3/2} dx) = \frac{5}{4} \underbrace{\int_0^1 x^{3/2} dx}_{2/5} = \frac{1}{2}.$$



For path C_2 , we break it into two segments, the first between $(0, 0)$ and $(0, 1)$ where $dx = 0$, and the second between $(0, 1)$ and $(1, 1)$ where $dy = 0$. Thus, equation (1) becomes:

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} (xy dx + \frac{1}{2}x^2 dy) = \frac{1}{2} \int_0^1 x^2 dy \Big|_{x=0} + \int_0^1 xy dx \Big|_{y=1} = \int_0^1 x dx = \frac{1}{2} = \int_{C_1} \vec{F} \cdot d\vec{r},$$

confirming that \vec{F} is conservative.

b) Next, consider $\vec{F} = xy\hat{i} + x^2\hat{j}$. For path C_1 , $y = x^{1/2}$, $dy = \frac{1}{2}x^{-1/2}dx$, and equation (1) becomes:

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 (x^{3/2} dx + \frac{1}{2}x^{3/2} dx) = \frac{3}{2} \underbrace{\int_0^1 x^{3/2} dx}_{2/5} = \frac{3}{5}.$$

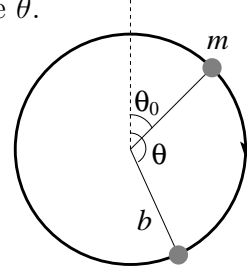
Conversely, along the two segments of path C_2 , equation (1) becomes:

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} (xy dx + x^2 dy) = \int_0^1 x^2 dy \Big|_{x=0} + \int_0^1 xy dx \Big|_{y=1} = \int_0^1 x dx = \frac{1}{2} \neq \int_{C_1} \vec{F} \cdot d\vec{r},$$

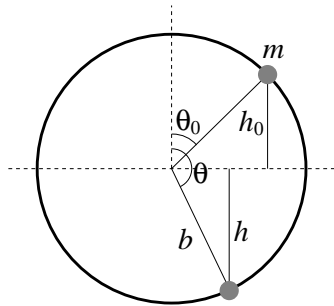
and \vec{F} as given in this part is *not* conservative.

Problem 3 (FC 4.22) In the diagram, a bead of mass m slides on a smooth circular wire of radius b starting from rest at position angle, θ_0 , to position angle θ .

- Find the speed of m at θ .
- Find the normal force exerted by the wire on m at θ .
- Find the values of θ_0 and θ where N a maximum, and find N_{\max} .



Solution: a) Conserve energy:

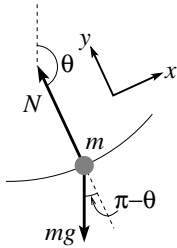


$$E = mgh_0 = mgb \cos \theta_0$$

$$= mgh + \frac{1}{2}mv^2 = mgb \cos \theta + \frac{1}{2}mv^2$$

$$\Rightarrow \boxed{v = \sqrt{2gb(\cos \theta_0 - \cos \theta)}}. \quad (1)$$

b) From the FBD and using equation (1),



$$N - mg \underbrace{\cos(\pi - \theta)}_{-\cos \theta} = ma_{\text{cent}} = \frac{mv^2}{b}$$

$$\Rightarrow N = 2mg(\cos \theta_0 - \cos \theta) - mg \cos \theta$$

$$\Rightarrow \boxed{N = mg(2 \cos \theta_0 - 3 \cos \theta)}. \quad (2)$$

c) We can see from equation (2) that N is maximised when $\theta_0 = 0$ (starts from top) and when $\theta = \pi$ (bead gets to bottom), at which point:

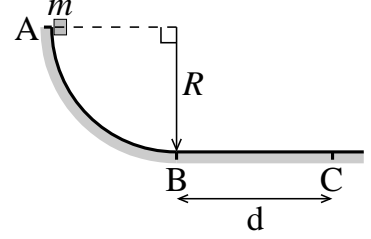
$$\boxed{N_{\max} = 5mg}.$$

Problem 4 The figure shows a track consisting of a quarter circle of radius R attached seamlessly at point B to a straight portion of indefinite length. A mass, m , is released from rest at point A , slides down the track to B , and comes to rest at C , a distance d from B .

If the coefficient of kinetic friction between m and the entire track is μ_k , use the W-K theorem to find the *average* frictional force along the curved portion of the track, AB. Recall that by definition, the average value of a quantity, q , over a path P of length L is,

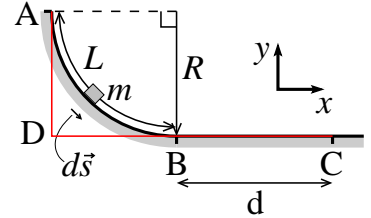
$$\langle q \rangle_P \equiv \frac{1}{L} \int_P q(s) ds, \quad (1)$$

where ds is an infinitesimal displacement along the path.



Solution: To use the W-K theorem, we need expressions for the work done by each of the forces acting on m as it slides down the track, namely gravity, mg , the normal force from the track, N , and the kinetic friction force from the track, f_k .

Since gravity is conservative, W_{mg} is path independent and we need not evaluate it along the path m actually takes (ABC). Instead, consider the red path ADC in the figure with a vertical drop R followed by a horizontal translation DC. Along this path, mg does work only along AD, which is much easier to compute than over ABC:



$$W_{mg} = mgR. \quad (2)$$

Since $W_g > 0$, gravity acts to accelerate m .

As always, the normal force, \vec{N} , does no work leaving only the work done by friction, which we'll find by considering the track in two pieces. Along AB where $f_k = \mu_k N$ is a variable force, and,

$$W_{f_k, AB} = \int_A^B \vec{f}_k \cdot d\vec{s} = - \underbrace{\int_A^B f_k ds}_{\langle f_k \rangle_{AB} L} = - \langle f_k \rangle_{AB} \frac{\pi}{2} R, \quad (3)$$

using equation (1) and the fact that \vec{f}_k is everywhere antiparallel to an infinitesimal displacement, $d\vec{s}$ along AB (see figure). Further, the path length, L , is a quarter of the circumference of a circle of radius R , and thus $\pi R/2$.

Along BC, where $f_k = \mu_k N = \mu_k mg = \text{constant}$, we have:

$$W_{f_k, BC} = \int_B^C \vec{f}_k \cdot d\vec{s} = -\mu_k mg \int_B^C dx = -\mu_k mgd. \quad (4)$$

Thus, from the W-K theorem and equations (2)–(4), we can write:

$$\begin{aligned} \sum W &= W_{mg} + W_N + W_{f_k, AB} + W_{f_k, BC} = mgR - \langle f_k \rangle_{AB} \frac{\pi}{2} R - \mu_k mgd \\ &= \Delta K = K_C - K_A = 0 \end{aligned}$$

$$\Rightarrow \boxed{\langle f_k \rangle_{AB} = \frac{2}{\pi} mg \left(1 - \mu_k \frac{d}{R} \right)},$$

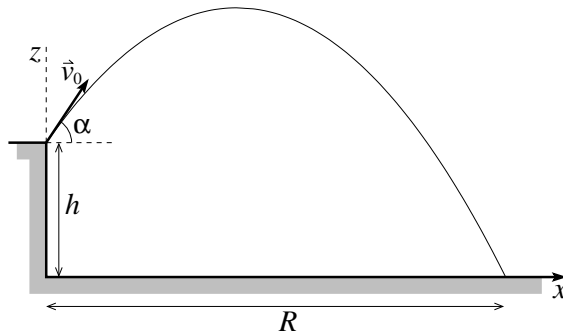
is the average friction force over the curved portion of the path.

Problem 5 (FC 4.9) A cannon is situated at the edge of a bluff a height h over a plain below, and fires a cannon ball with a muzzle speed v_0 .

- a) Ignoring air resistance, show that the elevation angle, α , required to achieve maximum range across the plain is given by:

$$\sin^2 \alpha = \frac{1}{2} \frac{v_0^2}{v_0^2 + gh}.$$

- b) What is the maximum range, R_{\max} , of the cannon in terms of v_0 , h , and g ?



Solution: We start with equation 4.3.9 (in ed. 7) giving the trajectory of a projectile in the absence of air resistance, and generalised for a non-zero starting height, z_0 :

$$z(x) - z_0 = x \tan \alpha - \frac{g}{2v_0^2 \cos^2 \alpha} x^2. \quad (1)$$

For $z_0 = h$, the range, R , is the value of x when $z = 0$ (see figure). Thus, for a given α ,

$$\begin{aligned} z(R) = 0 &= h + R \tan \alpha - \frac{g}{2v_0^2 \cos^2 \alpha} R^2 \\ \Rightarrow R^2 - \frac{2v_0^2 R}{g} \tan \alpha \cos^2 \alpha - \frac{2v_0^2 h}{g} \cos^2 \alpha, \end{aligned}$$

a quadratic in R . Since $2 \tan \alpha \cos^2 \alpha = 2 \sin \alpha \cos \alpha = \sin 2\alpha$ and $2 \cos^2 \alpha = \cos 2\alpha + 1$,

$$R^2 - R \frac{v_0^2}{g} \sin 2\alpha - \frac{v_0^2 h}{g} (\cos 2\alpha + 1) = 0, \quad (2)$$

To find the angle, α , that extremises R , we need to set $dR/d\alpha = 0$. To do that, we *could* first solve Eq. (2) for R using the quadratic formula:

$$R = \frac{v_0^2 \sin 2\alpha}{2g} \pm \sqrt{\frac{v_0^4 \sin^2 2\alpha}{4g^2} + \frac{v_0^2 h}{g} (\cos 2\alpha + 1)},$$

and *then* find $dR/d\alpha$, but this looks awful!! So instead, let's differentiate Eq. (2) *implicitly*:

$$2R \frac{dR}{d\alpha} - \frac{dR}{d\alpha} \frac{v_0^2}{g} \sin 2\alpha - R \frac{v_0^2}{g} 2 \cos 2\alpha + \frac{v_0^2 h}{g} 2 \sin 2\alpha = 0$$

$$\Rightarrow -R \cos 2\alpha + h \sin 2\alpha = 0 \Rightarrow R = h \tan 2\alpha. \quad (3)$$

Substituting equation (3) into equation (2), we get:

$$\begin{aligned} h^2 \tan^2 2\alpha - \mathcal{K} \tan 2\alpha \frac{v_0^2}{g} \sin 2\alpha - \frac{v_0^2 \mathcal{K}}{g} (\cos 2\alpha + 1) &= 0 \\ \Rightarrow h \frac{\sin^2 2\alpha}{\cos^2 2\alpha} - \frac{v_0^2}{g} \frac{\sin^2 2\alpha}{\cos 2\alpha} - \frac{v_0^2}{g} (\cos 2\alpha + 1) &= 0, \end{aligned}$$

eliminating the tan function. Next, putting terms proportional to v_0^2/g on the RHS, we get,

$$\begin{aligned} h \frac{\sin^2 2\alpha}{\cos^2 2\alpha} &= \frac{v_0^2}{g} \left(\frac{\sin^2 2\alpha}{\cos 2\alpha} + \cos 2\alpha + 1 \right) = \frac{v_0^2}{g} \frac{\sin^2 2\alpha + \cos^2 2\alpha + \cos 2\alpha}{\cos 2\alpha} \\ \Rightarrow h \frac{\sin^2 2\alpha}{\cos^2 2\alpha} &= \frac{v_0^2}{g} \frac{1 + \cos 2\alpha}{\cos 2\alpha}. \end{aligned}$$

Then, use identities: $\sin 2\alpha = 2 \sin \alpha \cos \alpha$; $\cos 2\alpha = 1 - \sin^2 \alpha$; $1 + \cos 2\alpha = \cos^2 \alpha$, to get:

$$\begin{aligned} h \frac{4 \sin^2 \alpha \cos^2 \alpha}{1 - 2 \sin^2 \alpha} &= \frac{v_0^2}{g} \frac{2 \cos^2 \alpha}{1 - 2 \sin^2 \alpha} \Rightarrow \frac{gh}{v_0^2} = \frac{1 - 2 \sin^2 \alpha}{2 \sin^2 \alpha} = \frac{1}{2 \sin^2 \alpha} - 1 \\ \Rightarrow \frac{1}{2 \sin^2 \alpha} &= 1 + \frac{gh}{v_0^2} = \frac{v_0^2 + gh}{v_0^2} \Rightarrow \boxed{\sin^2 \alpha = \frac{1}{2} \frac{v_0^2}{v_0^2 + gh}}, \end{aligned} \quad (4)$$

as desired.

b) To find R , we first note from Eq. (4) that:

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha = 1 - \frac{v_0^2}{v_0^2 + gh} = \frac{gh}{v_0^2 + gh}.$$

Then, from Eq. (3), we have:

$$\begin{aligned} R_{\max} &= h \tan 2\alpha = h \frac{\sin 2\alpha}{\cos 2\alpha} = h \frac{\sqrt{1 - \cos^2 2\alpha}}{\cos 2\alpha} = \mathcal{K} \sqrt{1 - \left(\frac{gh}{v_0^2 + gh} \right)^2} \frac{v_0^2 + gh}{g\mathcal{K}} \\ &= \frac{1}{g} \sqrt{(v_0^2 + gh)^2 - (gh)^2} = \frac{1}{g} \sqrt{v_0^4 + 2v_0^2 gh} \\ \Rightarrow \boxed{R_{\max} &= \frac{v_0^2}{g} \sqrt{1 + \frac{2gh}{v_0^2}}}. \end{aligned}$$

Note that for $h = 0$, this reduces further to $R_{\max} = v_0^2/g$, the range worked out in PHYS 1210 for a cannon on the same level as the target.

Problem 6 (Challenge problem, 10 point bonus) A track with coefficient of kinetic friction μ_k over its entire length consists of a quarter circle of radius R attached seamlessly at point B to a straight portion of arbitrary length. A mass, m , is placed on the curved portion at point A such that angle $AOB = \theta_0$, and released from rest. It slides down the track past point B , and comes to rest at point C , a distance d beyond B .

- a) From Newton's 2nd Law, show that the centripetal acceleration, a_{cp} , of m as it slides along the quarter circle is given by the 1st order ODE:

$$\frac{da_{cp}}{d\theta} - 2\mu_k a_{cp} = 2g(\mu_k \cos \theta - \sin \theta), \quad (1)$$

where $\theta_0 > \theta > 0$ is the angle mOB .

Hint: You may find it useful to know that the tangential acceleration and centripetal acceleration can be related as follows:

$$a_{tan} = \frac{dv_{tan}}{dt} = \frac{d\theta}{dt} \frac{dv_{tan}}{d\theta} = -\frac{v_{tan}}{R} \frac{dv_{tan}}{d\theta} = -\frac{1}{2} \frac{d}{d\theta} \left(\frac{v_{tan}^2}{R} \right) = -\frac{1}{2} \frac{da_{cp}}{d\theta}.$$

Here, I've used $\dot{\theta} = -\frac{v_{tan}}{R}$ since v_{tan} increases in the direction of decreasing θ .

- b) Solve equation (1) for $a_{cp}(\theta)$ by finding the homogeneous solution, adding on a particular solution, and then applying the boundary condition (at $\theta = \theta_0$, $a_{cp} = 0$). After some algebra, show that at B ($\theta = 0$):

$$v_B^2 = a_{cp,B} R = \frac{2gR}{1 + 4\mu_k^2} \left[(1 - 2\mu_k^2)(1 - \cos \theta_0 e^{-2\mu_k \theta_0}) - 3\mu_k \sin \theta_0 e^{-2\mu_k \theta_0} \right] \quad (2)$$

- c) Use the W-K theorem between B and C to find the distance, d , where m comes to rest.

Solution: a) Consider the FBD for m when it is somewhere between points A and B , and thus $0 < \theta < \theta_0$. From Newton's 2nd Law, we have:

$$x/ \quad -f_k + mg \sin \theta = ma_x = ma_{tan}; \quad (3)$$

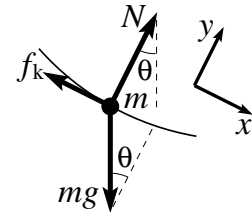
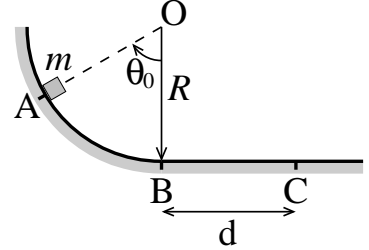
$$y/ \quad N - mg \cos \theta = ma_y = m \frac{v^2}{R} = ma_{cp}, \quad (4)$$

where, since path AB is circular, a_x is a tangential acceleration, and a_y is a centripetal acceleration. Then, from the hint,

$$a_{tan} = -\frac{1}{2} \frac{da_{cp}}{d\theta}.$$

Substituting this along with $f_k = \mu_k N$ into equation (3) yields:

$$-\mu_k N + mg \sin \theta = -m \frac{1}{2} \frac{da_{cp}}{d\theta} \Rightarrow N = \frac{m}{\mu_k} \left(g \sin \theta + \frac{1}{2} \frac{da_{cp}}{d\theta} \right),$$



and substituting this into equation (4) gives us a first order differential equation for a_{cp} :

$$\begin{aligned} \frac{\mathfrak{M}}{\mu_{\text{k}}} \left(g \sin \theta + \frac{1}{2} \frac{da_{\text{cp}}}{d\theta} \right) - \mathfrak{M}g \cos \theta &= \mathfrak{M}a_{\text{cp}} \\ \Rightarrow \boxed{\frac{da_{\text{cp}}}{d\theta} - 2\mu_{\text{k}}a_{\text{cp}} = 2g(\mu_{\text{k}} \cos \theta - \sin \theta)}, \end{aligned}$$

in agreement with equation (1).

b) To solve equation (1), first consider the *homogeneous* equation:

$$\frac{da_{\text{cp,h}}}{d\theta} - 2\mu_{\text{k}}a_{\text{cp,h}} = 0,$$

whose solution we can write down by inspection:

$$a_{\text{cp,h}} = \alpha e^{2\mu_{\text{k}}\theta},$$

where α is a constant of integration. Alternately, one could recognise the homogeneous equation as a first order separable ODE, and solve directly to get the same result.

Next, we seek a particular solution to equation (1). Again by inspection, we see that since the RHS has both a cosine and a sine term, a particular solution must have the form:

$$a_{\text{cp,p}} = \beta \cos \theta + \gamma \sin \theta,$$

which we substitute into equation (1) directly to solve for β and γ . Thus,

$$\begin{aligned} -\beta \sin \theta + \gamma \cos \theta - 2\mu_{\text{k}}(\beta \cos \theta + \gamma \sin \theta) &= 2g(\mu_{\text{k}} \cos \theta - \sin \theta) \\ \Rightarrow (-\beta - 2\mu_{\text{k}}\gamma + 2g) \sin \theta &= (2g\mu_{\text{k}} - \gamma + 2\mu_{\text{k}}\beta) \cos \theta. \end{aligned}$$

Since $\cos \theta$ and $\sin \theta$ are linearly independent functions, the coefficients must be zero:

$$\left. \begin{aligned} -\beta - 2\mu_{\text{k}}\gamma + 2g &= 0 \\ 2g\mu_{\text{k}} - \gamma + 2\mu_{\text{k}}\beta &= 0 \end{aligned} \right\} \Rightarrow \beta = 2g \frac{1 - 2\mu_{\text{k}}^2}{1 + 4\mu_{\text{k}}^2} \quad \text{and} \quad \gamma = 6g \frac{\mu_{\text{k}}}{1 + 4\mu_{\text{k}}^2}. \quad (5)$$

Thus, the solution to equation (1) is:

$$a_{\text{cp}} = a_{\text{cp,h}} + a_{\text{cp,p}} = \alpha e^{2\mu_{\text{k}}\theta} + \beta \cos \theta + \gamma \sin \theta. \quad (6)$$

To evaluate α , we apply initial conditions: at $t = 0$, $\theta = \theta_0$ and $a_{\text{cp}} = 0$ ($v = 0$). Thus,

$$0 = \alpha e^{2\mu_{\text{k}}\theta_0} + \beta \cos \theta_0 + \gamma \sin \theta_0 \quad \Rightarrow \quad \alpha = -(\beta \cos \theta_0 + \gamma \sin \theta_0) e^{-2\mu_{\text{k}}\theta_0}. \quad (7)$$

Substituting equation (7) into (6), we get:

$$a_{\text{cp}} = -(\beta \cos \theta_0 + \gamma \sin \theta_0) e^{2\mu_{\text{k}}(\theta - \theta_0)} + \beta \cos \theta + \gamma \sin \theta. \quad (8)$$

Now, at point B, $\theta = 0$ and equation (8) becomes:

$$a_{cp,B} = \frac{v_B^2}{R} = -(\beta \cos \theta_0 + \gamma \sin \theta_0)e^{-2\mu_k \theta_0} + \beta$$

$$\Rightarrow \boxed{v_B^2 = R \left[\beta(1 - \cos \theta_0 e^{-2\mu_k \theta_0}) - \gamma \sin \theta_0 e^{-2\mu_k \theta_0} \right]},$$

which agrees with equation (2) after equations (5) have been substituted in.

c) To find the distance d that m slides beyond point B and comes to rest at point C, we use the W-K theorem. Between points B and C, the only force that does work is $f_k = \mu_k mg$, and thus,

$$\sum W = \Delta K \quad \Rightarrow \quad -\mu_k mgd = \cancel{K_C}^0 - K_B = -\frac{1}{2}mv_B^2$$

$$\Rightarrow \quad d = \frac{1}{2\mu_k g}v_B^2 = \frac{R}{2\mu_k g} \left[\beta(1 - \cos \theta_0 e^{-2\mu_k \theta_0}) - \gamma \sin \theta_0 e^{-2\mu_k \theta_0} \right]$$

$$\Rightarrow \quad \boxed{d = \frac{R}{\mu_k(1 + 4\mu_k^2)} \left[(1 - 2\mu_k^2)(1 - \cos \theta_0 e^{-2\mu_k \theta_0}) - 3\mu_k \sin \theta_0 e^{-2\mu_k \theta_0} \right]},$$

using equations (5).
