Solutions to Assignment 7

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Problem 1 (FC 4.12) A baseball player, who can throw a ball horizontally more easily than vertically, throws a ball at speed $v_0 \cos(\alpha/2)$ where $v_0 = 25\ \mathrm{m\,s^{-1}}$ is his speed when thrown horizontally, and $0 < \alpha < \pi/2$ is the elevation angle.

a) Find the value of α that maximises the height, H, and evaluate $H_{\max}$.

b) Find the value of α that maximises the range, R, and evaluate $R_{\max}$.

Solution: The only difference between this problem and the discussion in §4.3 (ed. 7 is more extensive than ed. 6) is the additional factor of $\cos(\alpha/2)$ in the initial speed. Thus, for any result from §4.3 that we may wish to use, we only need to replace v_0 with $v_0 \cos(\alpha/2)$.

a) Accordingly, we modify equation 4.3.11 (new to ed. 7) giving the maximum height (using H instead of $z_{\max}$) to read:

$$H = \frac{v_0^2 \cos^2(\alpha/2)\sin^2\alpha}{2g} = \frac{v_0^2(1+\cos\alpha)\sin^2\alpha}{4g} \tag{1}$$

using the trig identity $\cos^2(\alpha/2) = \frac{1}{2}(1+\cos\alpha)$. To find the value of α that maximises H, set $\partial_\alpha H = 0$ and solve for α:

$$\partial_\alpha H = \frac{v_0^2}{4g}\big((1+\cos\alpha)2\sin\alpha\cos\alpha - \sin^3\alpha\big) = 0$$

$$\Rightarrow \quad \sin\alpha(3\cos^2\alpha + 2\cos\alpha - 1) = \sin\alpha(3\cos\alpha - 1)(\cos\alpha + 1) = 0.$$

Thus, there are three roots and three values of α that extremise H. The root $\sin\alpha = 0 \Rightarrow \alpha = 0$ minimises H, while the root $\cos\alpha = -1 \Rightarrow \alpha = -\pi$ falls outside the presumed range $0 < \alpha < \pi/2$, and is discarded. The third root,

$$\cos\alpha = 1/3 \quad \Rightarrow \quad \boxed{\alpha = 70.5^\circ,}$$

maximises H, and is the root we seek. Using $g = 9.81\ \mathrm{m\,s^{-2}}$, we find $H_{\max}$ by substituting v_0, g, and $\cos\alpha = \frac{1}{3}$ (and thus $\sin^2\alpha = \frac{8}{9}$) into equation (1) to get:

$$\boxed{H_{\max} = 18.9\,\mathrm{m}.}$$

b) Next, we modify (and correct) equation 4.3.14 (new to ed. 7) for the range (the sine-function should not be squared) to read:

$$R = \frac{v_0^2\cos^2(\alpha/2)\sin(2\alpha)}{g} = \frac{v_0^2}{g}\sin\alpha\cos\alpha(1+\cos\alpha). \tag{2}$$

To find the value of α that maximises R, set $\partial_\alpha R = 0$ and solve for α:

$$\partial_\alpha R \;=\; \frac{v_0^2}{g}\big(\cos^2\alpha(1+\cos\alpha) - \sin^2\alpha(1+\cos\alpha) - \sin^2\alpha\cos\alpha\big) \;=\; 0$$

$$\Rightarrow \quad 3\cos^3\alpha + 2\cos^2\alpha - 2\cos\alpha - 1 \;=\; 0, \tag{3}$$

after a little algebra. By inspection, we see that $\cos\alpha = -1$ is a root, and divide it out of the polynomial in equation (3) as follows:

$$\begin{array}{r|lllll}
\multicolumn{1}{r}{} & 3\cos^2\alpha & -\cos\alpha & -1 & \\
\cline{2-5}
\cos\alpha+1 & 3\cos^3\alpha & +\,2\cos^2\alpha & -\,2\cos\alpha & -\,1 \\
\multicolumn{1}{r}{} & 3\cos^3\alpha & +\,3\cos^2\alpha & & \\
\cline{2-3}
\multicolumn{1}{r}{} & & -\cos^2\alpha & -\,2\cos\alpha & \\
\multicolumn{1}{r}{} & & -\cos^2\alpha & -\cos\alpha & \\
\cline{3-4}
\multicolumn{1}{r}{} & & & -\cos\alpha & -\,1 \\
\multicolumn{1}{r}{} & & & -\cos\alpha & -\,1 \\
\cline{4-5}
\multicolumn{1}{r}{} & & & & 0
\end{array}$$

$$\Rightarrow \quad (\cos\alpha+1)(3\cos^2\alpha - \cos\alpha - 1) \;=\; 0.$$

Once again we have three roots:

$$\cos\alpha \;=\; -1 \quad \text{and} \quad \cos\alpha \;=\; \frac{1\pm\sqrt{1+12}}{6} \;=\; 0.7676\ldots \;\text{ or }\; -0.4343\ldots$$

The negative roots give angles outside the presumed range $0 < \alpha < \pi/2$ and are discarded, while the positive root,

$$\cos\alpha \;=\; 0.7676\ldots \quad \Rightarrow \quad \boxed{\alpha \;=\; 39.9^\circ,}$$

is the angle that maximises the range. Substituting numerical values for v_0 and g, and using $\alpha = 39.9^\circ$ in equation (2), we get:

$$\boxed{R_{\text{max}} \;=\; 55.4\,\text{m}.}$$

Problem 2 (FC 4.15) In class, we considered a projectile of mass m and velocity $\vec{v}$ experiencing a drag force given by $\vec{D} = -m\gamma\vec{v}$, where γ is the coefficient of linear air drag. By solving the differential equations stemming from Newton's 2nd Law, we found the projectile *path* to be:

$$\vec{r}(t) \;=\; \frac{1}{\gamma}\left[\left(\vec{v}_0 + \frac{g}{\gamma}\hat{k}\right)\left(1-e^{-\gamma t}\right) - gt\hat{k}\right], \tag{1}$$

where $\vec{v}_0$ is the projectile's initial velocity and $\hat{k}$ is a unit vector in the vertical (z) direction.

a) Show that the *trajectory* of the projectile is given by:

$$z(x) \;=\; \frac{x}{v_{0x}}\left(v_{0z}+\frac{g}{\gamma}\right)+\frac{g}{\gamma^2}\ln\left(1-\frac{\gamma x}{v_{0x}}\right), \tag{2}$$

where x is the horizontal direction along which the projectile travels, and $\vec{v}_0=(v_{0x},v_{0z})$.

b) Show that the range of the projectile, R (value of $x>0$ when $z=0$), is given by:

$$R \;=\; R_0\left(1-\frac{4\gamma v_0}{3g}\sin\alpha+\mathcal{O}(\gamma^2)\right),$$

where $R_0=(v_0^2\sin 2\alpha)/g$ is the range of the particle for $\gamma=0$ (no air drag), α is the inclination angle of the projectile at the beginning of its trajectory, and $\mathcal{O}(\gamma^2)$ represents ignored terms of power γ^2 or higher.

Solution: a) From equation (1), we can write out the x- and z-components separately:

$$x(t) \;=\; \frac{v_{0x}}{\gamma}\left(1-e^{-\gamma t}\right); \tag{3}$$

$$z(t) \;=\; \frac{1}{\gamma}\left(v_{0z}+\frac{g}{\gamma}\right)\left(1-e^{-\gamma t}\right)-\frac{gt}{\gamma}. \tag{4}$$

From equation (3), we have:

$$1-e^{-\gamma t} \;=\; \frac{\gamma x}{v_{0x}} \tag{5}$$

$$\Rightarrow\quad e^{-\gamma t} \;=\; 1-\frac{\gamma x}{v_{0x}} \quad\Rightarrow\quad t \;=\; -\frac{1}{\gamma}\ln\left(1-\frac{\gamma x}{v_{0x}}\right). \tag{6}$$

Substituting equation (5) into the first term of equation (4) and equation (6) into the second term, we get:

$$\boxed{z(x) \;=\; \frac{1}{\cancel{\gamma}}\left(v_{0z}+\frac{g}{\gamma}\right)\frac{\cancel{\gamma}x}{v_{0x}}+\frac{g}{\gamma^2}\ln\left(1-\frac{\gamma x}{v_{0x}}\right),}$$

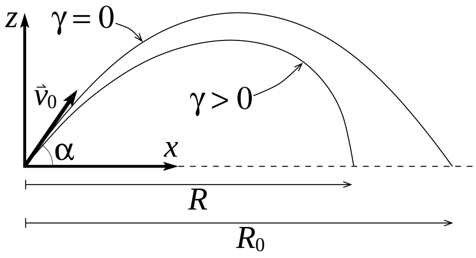
as desired.

b) To find the range, R, as indicated in the figure, set $z(R)=0$ in equation (2). Thus,

$$\frac{R}{v_{0x}}\left(v_{0z}+\frac{g}{\gamma}\right)+\frac{g}{\gamma^2}\ln\left(1-\frac{\gamma R}{v_{0x}}\right) \;=\; 0. \tag{7}$$

To make further progress, do a Maclaurin expansion on the logarithm:

$$\ln(1+u) \;=\; u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\ldots$$



Thus, for small γ ($\gamma \ll v_0/g$), equation (7) becomes:

$$\frac{R}{v_{0x}}\left(v_{0z} + \frac{g}{\gamma}\right) + \frac{g}{\gamma^2}\left(-\frac{\gamma R}{v_{0x}} - \frac{(\gamma R)^2}{2v_{0x}^2} - \frac{(\gamma R)^3}{3v_{0x}^3} - \dots\right)$$

$$= \frac{v_{0z}}{v_{0x}}R + \cancel{\frac{Rg}{v_{0x}\gamma}} - \cancel{\frac{gR}{\gamma v_{0x}}} - \frac{gR^2}{2v_{0x}^2} - \frac{g\gamma R^3}{3v_{0x}^3} - \dots \;=\; 0$$

$$\Rightarrow \quad v_{0z} - \frac{gR}{2v_{0x}} - \frac{g\gamma R^2}{3v_{0x}^2} - \mathcal{O}(\gamma^2) \;=\; 0,$$

having multiplied through by v_{0x}/R. This is as far as we got in class.

Next, drop terms of $\mathcal{O}(\gamma^2)$ to leave a quadratic in R:

$$R^2 + \frac{3v_{0x}}{2\gamma}R - \frac{3v_{0x}^2 v_{0z}}{g\gamma} \;\approx\; 0,$$

on which we use the quadratic formula to get:

$$R \;=\; -\frac{3v_{0x}}{4\gamma} \pm \sqrt{\frac{9v_{0x^2}}{16\gamma^2} + \frac{3v_{0x}^2 v_{0z}}{g\gamma}} \;=\; -\frac{3v_{0x}}{4\gamma} + \frac{3v_{0x}}{4\gamma}\sqrt{1 + \frac{16\gamma v_{0z}}{3g}},$$

keeping the + root so that $R > 0$. Next, apply a binomial expansion to the radical to get:

$$R \;=\; -\cancel{\frac{3v_{0x}}{4\gamma}} + \frac{3v_{0x}}{4\gamma}\left(\cancel{1} + \frac{1}{2}\,\frac{16\gamma v_{0z}}{3g} - \frac{1}{8}\,\frac{256\gamma^2 v_{0z}^2}{9g^2} + \mathcal{O}(\gamma^3)\right)$$

$$= \frac{2v_{0x}v_{0z}}{g} - \frac{8\gamma v_{0x}v_{0z}^2}{3g^2} + \mathcal{O}(\gamma^2) \;=\; \frac{2v_{0x}v_{0z}}{g}\left(1 - \frac{4\gamma v_{0z}}{3g} + \mathcal{O}(\gamma^2)\right).$$

On substituting $v_{0x} = v_0\cos\alpha$ and $v_{0z} = v_0\sin\alpha$, we get:

$$R \;=\; \frac{2v_0^2\sin\alpha\cos\alpha}{g}\left(1 - \frac{4\gamma v_0}{3g}\sin\alpha + \mathcal{O}(\gamma^2)\right) \;=\; \boxed{R_0\left(1 - \frac{4\gamma v_0}{3g}\sin\alpha + \mathcal{O}(\gamma^2)\right),}$$

as desired (having used the trig identity $2\sin\alpha\cos\alpha = \sin 2\alpha$).

Problem 3 (FC 4.17 modified) A mass m is suspended by six massless springs aligned along the Cartesian axes (*e.g.*, Fig. 4.4.1 of eds. 6 and 7) with spring constants $(k_x, k_y, k_z) = k\left(\frac{9}{4}, 1, 4\right)$, where $k = \pi^2 m/2$ (and thus, with two springs in the y-direction, for example, $F_y = -2k_y y = -\pi^2 m y$).

a) If, at $t = 0$, $\vec{r} = (1, -1, 1)$ and $\vec{v} = \vec{0}$, find $\vec{r}(t)$.

b) Does m ever retrace its path and, if so, find the smallest value of t where m returns to its initial conditions.

Solution: a) According to equation 4.4.19, the motion of the non-isotropic oscillator is given by:

$$\begin{aligned} x(t) &= A\cos(\omega_x t + \alpha) &\Rightarrow\quad v_x(t) &= -A\omega_x \sin(\omega_x t + \alpha); \\ y(t) &= B\cos(\omega_y t + \beta) &\Rightarrow\quad v_y(t) &= -B\omega_y \sin(\omega_y t + \beta); \\ z(t) &= C\cos(\omega_z t + \gamma) &\Rightarrow\quad v_z(t) &= -C\omega_z \sin(\omega_z t + \gamma), \end{aligned}$$

where the frequencies are given by:

$$\omega_x = \sqrt{\frac{2k_x}{m}} = \frac{3\pi}{2}; \qquad \omega_y = \sqrt{\frac{2k_y}{m}} = \pi; \qquad \omega_z = \sqrt{\frac{2k_z}{m}} = 2\pi.$$

and the phases α, β, and γ are determined from initial conditions.

First, from $\vec{v}(0) = \vec{0}$,

$$-A\omega_x \sin\alpha = -B\omega_y \sin\beta = -C\omega_z \sin\gamma = 0 \quad\Rightarrow\quad \alpha = \beta = \gamma = 0.$$

Then, from $\vec{r}(0) = (1, -1, 1)$,

$$A\cos 0 = 1; \quad B\cos 0 = -1; \quad C\cos 0 = 1 \quad\Rightarrow\quad A = -B = C = 1,$$

and we have,

$$\boxed{x(t) = \cos\left(\frac{3\pi}{2}t\right); \qquad y(t) = -\cos(\pi t); \qquad z(t) = \cos(2\pi t).}$$

b) Since:

$$\frac{2\omega_x}{3} = \omega_y = \frac{\omega_z}{2} = \pi \quad\Rightarrow\quad \frac{\omega_x}{3} = \frac{\omega_y}{2} = \frac{\omega_z}{4} = \frac{\pi}{2},$$

the frequencies are *commensurate* and m traces out a *Lissajous figure* (a closed path), returning periodically to its initial conditions. Since the integers in the denominators have no common factors (2 and 4 have a common factor, namely 2, but 2 is not a factor of 3), the time required for m to complete a single loop and return to its initial conditions, τ, is given by equation 4.4.20 in the text:

$$\boxed{\tau = 2\pi\frac{3}{\omega_x} = 2\pi\frac{2}{\omega_y} = 2\pi\frac{4}{\omega_z} = 4, \quad \text{in units where } k = \pi^2 m/2.}$$

Problem 4 (FC 4.18) For a 2-D isotropic oscillator, we showed in class that the path taken by a mass, m, is an ellipse, given by:

$$\frac{x^2}{A^2} - xy\frac{2\cos\Delta}{AB} + \frac{y^2}{B^2} = \sin^2\Delta, \tag{1}$$

(*e.g.*, equation 4.4.10 in ed. 7 of the text), where A, B, and Δ were defined in class.

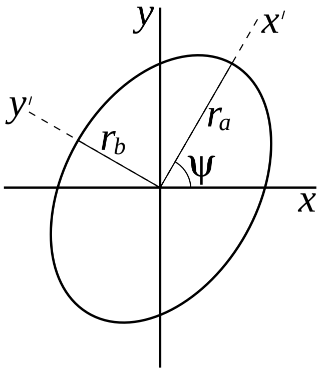
Show that the major axis of this ellipse is inclined relative to the x-axis by an angle ψ, where:

$$\tan 2\psi \;=\; 2\frac{AB\cos\Delta}{A^2 - B^2}.$$

Solution: The general expression for an ellipse, as shown in the figure, is:

$$ax^2 + bxy + cy^2 \;=\; f, \tag{2}$$

where the axes of the ellipse are aligned with the x-y axes ($\psi = 0$) only if $b = 0$.



Thus, define the x' and y' axes to be aligned with the axes of the ellipse, and note that the x-y axes are simply the x'-y' axes rotated by an angle $-\psi$. Thus,

$$\left.\begin{aligned} x &= x'\cos(-\psi) + y'\sin(-\psi) \;=\; x'\cos\psi - y'\sin\psi; \\ y &= -x'\sin(-\psi) + y'\cos(-\psi) \;=\; x'\sin\psi + y'\cos\psi. \end{aligned}\right\} \tag{3}$$

Substituting equations (3) into (2), we get:

$$a(x'\cos\psi - y'\sin\psi)^2 + b(x'\cos\psi - y'\sin\psi)(x'\sin\psi + y'\cos\psi) \\ + c(x'\sin\psi + y'\cos\psi)^2 \;=\; f$$

$$\Rightarrow \quad x'^2(\sim) + x'y'\big(-2a\cos\psi\sin\psi + b(\cos^2\psi - \sin^2\psi) + 2c\sin\psi\cos\psi\big) + y'^2(\sim) \;=\; f.$$

For the ellipse to align with the x'-y' axes, the cross-term vanishes and we have:

$$b(\cos^2\psi - \sin^2\psi) + (c-a)2\sin\psi\cos\psi \;=\; b\cos 2\psi - (a-c)\sin 2\psi \;=\; 0$$

$$\Rightarrow \quad \tan 2\psi \;=\; \frac{b}{a-c}. \tag{4}$$

Comparing equations (1) and (2), we have:

$$a \;=\; \frac{1}{A^2}; \qquad b \;=\; -\frac{2\cos\Delta}{AB}; \qquad c \;=\; \frac{1}{B^2}.$$

Substituting these into equation (4), we get the desired result:

$$\tan 2\psi \;=\; -\frac{2\cos\Delta}{AB}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)^{-1} \;=\; \frac{2\cos\Delta}{\cancel{AB}}\,\frac{(AB)^{\cancel{2}}}{A^2 - B^2} \;=\; \boxed{\frac{2AB\cos\Delta}{A^2 - B^2}.}$$

Problem 5 (FC 4.20) An electron moves in the electromagnetic field $\vec{E} = E\hat{\jmath}$, $\vec{B} = B\hat{k}$, where E and B are constant. If $\vec{r}_0 = 0$ and $\vec{v}_0 = v_0\hat{\imath}$, show that the resulting particle path is a cycloid parameterised by t and given by:

$$x(t) \;=\; a\sin\omega t + bt; \qquad y(t) \;=\; a(1-\cos\omega t); \qquad z(t) \;=\; 0,$$

where $\omega = eB/m$, $b = E/B$, and $a = (v_0 - b)/\omega$.

Solution: Let the position and velocity of the electron be $\vec{r} = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$ and $\vec{v} = \dot{x}\hat{\imath}+\dot{y}\hat{\jmath}+\dot{z}\hat{k}$. For an electron, $q = -e$, and the Lorentz force is:

$$\begin{aligned}\vec{F} \;&=\; -e(\vec{E}+\vec{v}\times\vec{B}) \;=\; -e\big(E\hat{\jmath} + \dot{x}B\,\underbrace{\hat{\imath}\times\hat{k}}_{-\hat{\jmath}} + \dot{y}B\,\underbrace{\hat{\jmath}\times\hat{k}}_{\hat{\imath}}\big) \;=\; -e\dot{y}B\hat{\imath} + e(\dot{x}B - E)\hat{\jmath}\\ &=\; m(\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath}+\ddot{z}\hat{k})\end{aligned}$$

$$\Rightarrow \quad \ddot{x} \;=\; -\frac{eB}{m}\dot{y} \;\equiv\; -\omega\dot{y}; \qquad \ddot{y} \;=\; \omega\dot{x} - \frac{eE}{m}; \qquad \ddot{z} \;=\; 0, \tag{1}$$

where $\omega = \dfrac{eB}{m}$ is the cyclotron frequency. Integrate over time the first of equations (1) to get:

$$\dot{x} \;=\; -\omega y + c,$$

where c is the constant of integration. Now, at $t = 0$, $y = 0 \;\Rightarrow\; \dot{x}_0 = v_0 = c$. Thus,

$$\dot{x} \;=\; -\omega y + v_0. \tag{2}$$

Substitute (2) into the second of equations (1) to get:

$$\ddot{y} \;=\; -\omega^2 y + \omega v_0 - \frac{eE}{m} \quad\Rightarrow\quad \ddot{y} + \omega^2 y \;=\; \omega v_0 - \frac{eE}{m}. \tag{3}$$

To solve (3), start with the *homogeneous* equation: $\ddot{y}_{\mathrm{H}} + \omega^2 y_{\mathrm{H}} = 0$, whose solution is:

$$y_{\mathrm{H}}(t) \;=\; -a\cos(\omega t + \delta),$$

where a and δ are constants of integration, and where we have inserted the minus sign in anticipation of the final result.

Next, a *particular* solution [*anything* that solves (3)] is evidently:

$$y_{\mathrm{P}} \;=\; \frac{v_0}{\omega} - \frac{eE}{m\omega^2} \;=\; \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right).$$

Thus, the general solution to (3) is:

$$y(t) \;=\; y_{\mathrm{H}}(t) + y_{\mathrm{P}} \;=\; -a\cos(\omega t+\delta) + \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right). \tag{4}$$

Now apply boundary conditions to (4). Thus,

$$\begin{aligned} \dot{y}(0) &= 0 \quad \Rightarrow \quad a\sin\delta = 0 \quad \Rightarrow \quad \delta = 0; \\ y(0) &= 0 \quad \Rightarrow \quad a\cos\delta = a = \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right), \end{aligned} \tag{5}$$

and (4) becomes:

$$y(t) = -a\cos\omega t + a = a(1 - \cos\omega t), \tag{6}$$

where a is given by (5). Next, substitute (6) into (2) to get:

$$\dot{x} = -\omega a(1 - \cos\omega t) + v_0,$$

which integrates to:

$$x(t) = -\omega a\left(t - \frac{1}{\omega}\sin\omega t\right) + v_0 t + \cancelto{0}{x_0} = a\sin\omega t + (v_0 - \omega a)t = a\sin\omega t + bt,$$

where $b = v_0 - \omega a = \dfrac{E}{B}$.

Finally, the last of equations (1) integrates trivially to $z(t) = z_0 + \dot{z}_0 t = 0$ since, from the given initial conditions, $z_0 = 0$ and $\dot{z}_0 = 0$.

Thus, the path of the electron, as parameterised by t, is given by:

$$\boxed{x(t) = a\sin\omega t + bt; \qquad y(t) = a(1 - \cos\omega t); \qquad z(t) = 0,}$$

where $\omega = eB/m$, $b = E/B$, and $a = (v_0 - b)/\omega$, as desired.