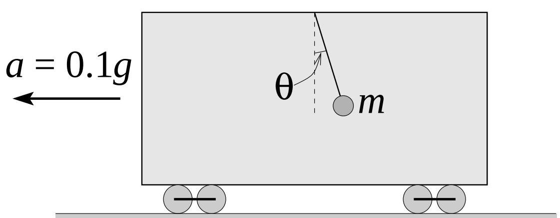
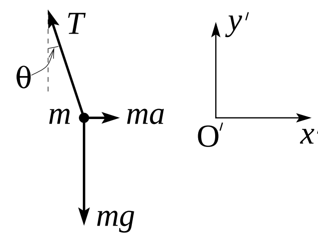
Solutions to Assignment 8

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1 (FC 5.3) A plumb bob hangs from the ceiling of a boxcar in a train accelerating at $g/10$. If the plumb line is held steady (not allowed to oscillate like a pendulum), find the tension in the cord.

Solution: As viewed from the O′ (accelerating) frame inside the boxcar, the FBD gives:





$$x'/ \quad -T\sin\theta + ma = 0 \quad \Rightarrow \quad T\sin\theta = ma = m\frac{g}{10} \tag{1}$$

$$y'/ \quad T\cos\theta - mg = 0 \quad \Rightarrow \quad T\cos\theta = mg \tag{2}$$

Squaring each of equations (1) and (2), then adding the result gives us:

$$T^2 = m^2\big((0.1g)^2 + g^2\big) = 1.01(mg)^2 \quad \Rightarrow \quad \boxed{T \simeq 1.005mg.}$$

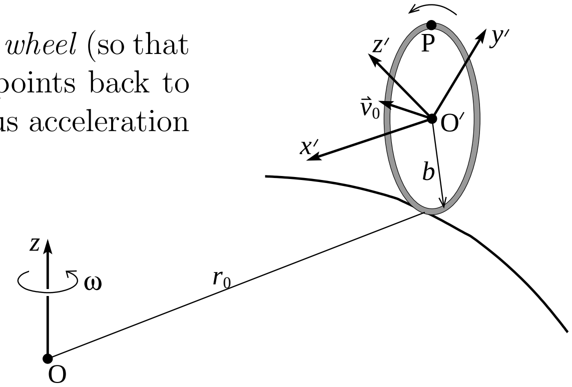
To find the direction, divide equation (1) by equation (2) to get:

$$\tan\theta = a/g = 0.1 \quad \Rightarrow \quad \boxed{\theta \simeq 5.7°.}$$

Problem 2 (FC 5.22) As in Example 4.3 from class (Example 5.2.2 in the text), a bicycle wheel of radius b travels with constant speed, v_0, around a track of radius r_0.

Using the coordinate system (x', y', z') *fixed to the wheel* (so that $\hat{\jmath}'$ and $\hat{k}'$ rotate with the wheel while $\hat{\imath}'$ always points back to $\hat{k}$ fixed to inertial frame O), find the instantaneous acceleration at point P, as shown in the figure.

This is different from what we did in class where $\hat{k}'$ remained vertical. If nothing else, this problem should convince you of the value in choosing a "sensible" coordinate system!



Hints:

1. As defined, all points along the rim are *at rest* relative to O′ $\Rightarrow$ $\vec{v}\,' = 0$ and $\vec{a}\,' = 0$.

2. If $\hat{\jmath}\,' \propto -\vec{v}_0$ and $\hat{k}'$ points vertically at $t = 0$, then at time t, $\hat{k}$ (unit vector in vertical direction) is given by:
$$\hat{k} \;=\; \sin\Omega t\,\hat{\jmath}\,' + \cos\Omega t\,\hat{k}',$$
where $\vec{\Omega} = (v_0/b)\,\hat{\imath}\,'$ is the angular velocity of the wheel about its own axis. Thus, when P reaches the highest point, its position vector relative to O′ is given by:
$$\vec{r}\,' \;=\; b\hat{k} \;=\; b\big(\sin\Omega t\,\hat{\jmath}\,' + \cos\Omega t\,\hat{k}'\big).$$

3. The most important thing to watch for is $\vec{\omega}$. Here, the angular velocity of O′ relative to O actually has *two* terms: one an "orbital" term as O′ goes around the track "orbiting" O at angular speed $\omega_{\rm orb} = v_0/r_0$, the other a "spin" term as O′ spins about its axis with angular speed Ω. Thus, $\vec{\omega} = \omega_{\rm orb}\hat{k} + \Omega\hat{\imath}\,'$ and, because $\hat{\imath}\,'$ is constantly changing direction, $\dot{\vec{\omega}} \neq 0$!!

Solution: With O′ at the axle and rotating with the wheel, $\hat{\imath}\,'$ always points towards the vertical axis at O, while $\hat{\jmath}\,'$ and $\hat{k}'$ point to fixed points on the rim of the wheel. Without loss of generality, suppose that at $t = 0$, $\hat{\jmath}\,' \propto -\vec{v}_0$ and $\hat{k}'$ points directly upward. Thus, at a time, t, later,
$$\hat{k} \;=\; \sin\Omega t\,\hat{\jmath}\,' + \cos\Omega t\,\hat{k}', \tag{1}$$
relative to the (x', y', z') coordinate system, and where $\vec{\Omega} = (v_0/b)\,\hat{\imath}\,'$ is the angular velocity of the wheel about its own axis.

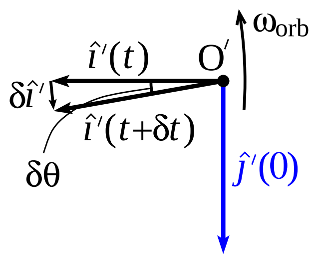
Step 1: "Hunting and gathering".

Consider now the specific point P fixed to the rim of the wheel, and thus motionless relative to O′. As it reaches the highest point above the ground,

$$\begin{aligned} \text{position of P relative to O}':&\qquad \vec{r}\,' \;=\; b\hat{k} \;=\; b\big(\sin\Omega t\,\hat{\jmath}\,' + \cos\Omega t\,\hat{k}'\big); \qquad (2)\\ \text{velocity of P relative to O}':&\qquad \vec{v}\,' \;=\; 0;\\ \text{acceleration of P relative to O}':&\qquad \vec{a}\,' \;=\; 0, \end{aligned}$$

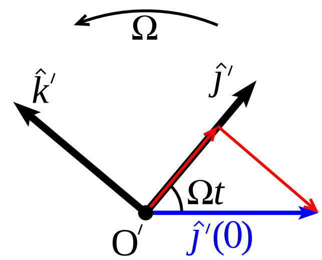
from hints 1 and 2. From hint 3, the angular velocity of O′ relative to O has both an "orbital" and "spin" component,
$$\vec{\omega} \;=\; \omega_{\rm orb}\hat{k} + \Omega\hat{\imath}\,' \;=\; \frac{v_0}{r_0}\,\hat{k} + \frac{v_0}{b}\hat{\imath}\,' \;=\; \frac{v_0}{r_0}\big(\sin\Omega t\,\hat{\jmath}\,' + \cos\Omega t\,\hat{k}'\big) + \frac{v_0}{b}\hat{\imath}\,', \tag{3}$$
using Eq. (1). Thus,
$$\dot{\vec{\omega}} \;=\; \frac{v_0}{b}\frac{d\hat{\imath}\,'}{dt}, \tag{4}$$
since direction of $\hat{\imath}\,'$ changes in time but not $\hat{k}$ (even when expressed in terms of $\hat{\jmath}\,'$ and $\hat{k}'$!).

As shown in the first inset (and we've done something very similar to this before in class), as the wheel goes around the track, the direction of the $\hat{\imath}'$ axis rotates from its direction at time t to its direction at time $t + \delta t$ with the difference vector, $\delta\hat{\imath}'$, given by,



$$\delta\hat{\imath}' \;=\; \underbrace{|\hat{\imath}'|}_{1}\,\delta\theta\,\hat{\jmath}'(0) \;=\; \omega_{\rm orb}\delta t\,\hat{\jmath}'(0),$$

where $\delta\theta = \omega_{\rm orb}\delta t$. Note that $\hat{\jmath}'(0)$ is the direction of $\hat{\jmath}'$ at $t = 0$ and is given by,



$$\hat{\jmath}'(0) \;=\; \cos\Omega t\,\hat{\jmath}' - \sin\Omega t\,\hat{k}',$$

where, as shown in the second inset, $\hat{\jmath}'(0)$ is the vector sum of the two red arrows, the one parallel to $\hat{\jmath}'$ having length $\cos\Omega t$, and the one *anti*parallel to $\hat{k}'$ having length $\sin\Omega t$. Thus,

$$\frac{d\hat{\imath}'}{dt} \;=\; \lim_{\delta t\to 0}\frac{\delta\hat{\imath}'}{\delta t} \;=\; \omega_{\rm orb}\big(\cos\Omega t\,\hat{\jmath}' - \sin\Omega t\,\hat{k}'\big)$$

$$\Rightarrow\quad \dot{\vec{\omega}} \;=\; \frac{v_0^2}{br_0}\big(\cos\Omega t\,\hat{\jmath}' - \sin\Omega t\,\hat{k}'\big), \tag{5}$$

using Eq. (4) and since $\omega_{\rm orb} = v_0/r_0$.

Finally, the acceleration of O′ relative to O is the centripetal acceleration is,

$$\vec{A} \;=\; \frac{v_0^2}{r_0}\,\hat{\imath}'. \tag{6}$$

Step 2: Do the cross products.

Using Eq. (5) and (2), we have,

$$\text{transverse term:}\qquad \dot{\vec{\omega}}\times\vec{r}\,' \;=\; \frac{v_0^2}{r_0}\big(\cos^2\Omega t(\hat{\jmath}'\times\hat{k}') - \sin^2\Omega t(\hat{k}'\times\hat{\jmath}')\big) \;=\; \frac{v_0^2}{r_0}\,\hat{\imath}';$$

$$\text{Coriolis term:}\qquad 2\vec{\omega}\times\vec{v}\,' \;=\; 0.$$

As for the centrifugal term, using Eq. (3) and (2), we get,

$$\begin{aligned}
\vec{\omega}\times\vec{r}\,' \;&=\; (\vec{\omega}_{\rm orb} + \vec{\Omega})\times\vec{r}\,' \;=\; \vec{\Omega}\times\vec{r}\,' \quad (\text{since } \vec{\omega}_{\rm orb}\parallel\vec{r}\,'\parallel\hat{k} \text{ at P})\\
&=\; v_0\big(\sin\Omega t(\hat{\imath}'\times\hat{\jmath}') + \cos\Omega t(\hat{\imath}'\times\hat{k}')\big) \;=\; v_0\big(\sin\Omega t\,\hat{k}' - \cos\Omega t\,\hat{\jmath}'\big)\\
\Rightarrow\quad \vec{\omega}\times(\vec{\omega}\times\vec{r}\,') \;&=\; (\vec{\omega}_{\rm orb} + \vec{\Omega})\times v_0\Big(\sin\Omega t\,\hat{k}' - \cos\Omega t\,\hat{\jmath}'\Big)\\
&=\; \frac{v_0^2}{r_0}\Big(\sin^2\Omega t(\hat{\jmath}'\times\hat{k}') - \cos^2\Omega t(\hat{k}'\times\hat{\jmath}')\Big)\\
&\qquad\qquad + \frac{v_0^2}{b}\Big(\sin\Omega t(\hat{\imath}'\times\hat{k}') - \cos\Omega t(\hat{\imath}'\times\hat{\jmath}')\Big)\\
&=\; \frac{v_0^2}{r_0}\,\hat{\imath}' - \frac{v_0^2}{b}\,\hat{k},
\end{aligned}$$

using Eq. (1) for $\hat{k}$.

Step 3: Assemble the accelerations: Using equation (5.2.14) of the text,

$$\vec{a} \;=\; \vec{a}\,' + \dot{\vec{\omega}} \times \vec{r}\,' + 2\,\vec{\omega} \times \vec{v}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') + \vec{A},$$

we find:

$$\vec{a} \;=\; 0 + \frac{v_0^2}{r_0}\,\hat{\imath}\,' + 0 + \frac{v_0^2}{r_0}\,\hat{\imath}\,' - \frac{v_0^2}{b}\hat{k} + \frac{v_0^2}{r_0}\,\hat{\imath}\,' \quad \Rightarrow \quad \boxed{\vec{a} \;=\; \frac{3v_0^2}{r_0}\,\hat{\imath}\,' - \frac{v_0^2}{b}\hat{k}.}$$

Note that our answer here seems to differ from the example done in class, where the last term was proportional to $\hat{k}'$, not $\hat{k}$. However, this difference is cosmetic since in that example where O$'$ was fixed to the axle but did not rotate with the wheel, $\hat{k}' = \hat{k}$. Here, $\hat{k}' \neq \hat{k}$, and we must use $\hat{k}$, not $\hat{k}'$ to indicate the vertical direction.

Note also the difference in how the terms are interpreted.

- In Example 4.3 from the class, $-(v_0^2/b)\hat{k}$ came from the acceleration of P relative to O$'$, which is zero here. Here this term is interpreted as part of the centrifugal acceleration, which was zero in the example.

- Also in Example 4.3, $2(v_0^2/r_0)\hat{\imath}\,'$ was the Coriolis acceleration. Here, the Coriolis acceleration is zero (since $\vec{v}\,' = 0$) and instead, a term $(v_0^2/r_0)\hat{\imath}\,'$ comes from each of the transverse term (zero in the example but non-zero here since $\dot{\vec{\omega}} \neq 0$) and a second centrifugal term.

- Finally, only the acceleration of O$'$ relative to O is the same in the example as here, both contributing $(v_0^2/r_0)\hat{\imath}\,'$.

This just goes to show that while the interpretation of the terms may be different from one frame of reference to another, the actual *measurable*—the acceleration $\vec{a}$ in this case—is unchanged.

Problem 3 (FC 5.8) A bug with mass m crawls with constant speed in a circular path of radius b about the centre of a turntable rotating with constant angular speed ω. If the coefficient of static friction between the bug and turntable surface is μ_s, how fast (relative to the turntable) can the bug crawl without slipping if the bug moves:

a) in the direction of rotation; and

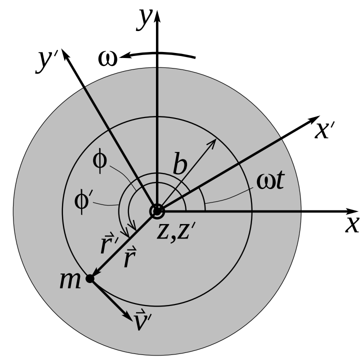
b) opposite to the direction of rotation?

Hint: Start with equation 5.3.2 (eds. 6 and 7) and, in the "hunting and gathering" step, be sure to include all *three* real forces.

Solution: a) Let the origins of both O and O' be at the centre of the turntable, and let O' rotate with the turntable. Then, with respect to O', we have:

$$\vec{F}' = \vec{F} - m\dot{\vec{\omega}} \times \vec{r}' - 2m\vec{\omega} \times \vec{v}' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') - m\vec{A}_0 = m\vec{a}', \tag{1}$$

where v' is the tangential speed of the bug in its circular path relative to O'. Then,



Step 1: "Hunting and gathering":

angular velocity of O' relative to O: $\vec{\omega} = \omega\hat{e}'_z$

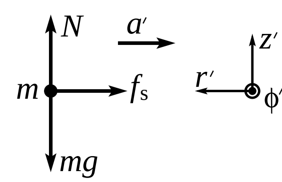
angular acceleration of O' relative to O: $\dot{\vec{\omega}} = 0;$

position of m relative to O': $\vec{r}' = b\hat{e}'_r;$

velocity of m relative to O': $\vec{v}' = v'\hat{e}'_\phi;$

acceleration of m relative to O': $\vec{a}' = -\dfrac{v'^2}{b}\hat{e}'_r;$

acceleration of O' relative to O: $\vec{A} = 0.$



Note that $\vec{a}'$ is the centripetal acceleration necessary for m to move in a circular path relative to O', and that $\vec{A} = 0$ since O and O' are coincident.

Step 2: Do the cross products:

transverse force: $-m\dot{\vec{\omega}} \times \vec{r}' = 0;$

Coriolis force: $-2m\vec{\omega} \times \vec{v}' = -2m\omega v'\hat{e}'_z \times \hat{e}'_\phi = 2m\omega v'\hat{e}'_r;$

centrifugal force: $-m\vec{\omega} \times (\vec{\omega} \times \vec{r}') = -\omega^2 b\,\hat{e}'_z \times (\hat{e}'_z \times \hat{e}'_r) = m\omega^2 b\hat{e}'_r.$

Step 3: Assemble the forces:

From the FBD, the real forces are: $\vec{F} = (N - mg)\hat{e}'_z - f_\mathrm{s}\hat{e}'_r$. Substituting everything into equation (1), we get:

$$\vec{F}' = (-f_\mathrm{s} + 2m\omega v' + m\omega^2 b)\hat{e}'_r + (N - mg)\hat{e}'_z = -m\frac{v'^2}{b}\hat{e}'_r,$$

which breaks up into r' and z' components as follows:

$$z'/ \quad F'_z = N - mg = 0 \quad \Rightarrow \quad N = mg;$$

$$r'/ \quad F'_r = -f_\mathrm{s} + 2m\omega v' + m\omega^2 b = -m\frac{v'^2}{b}.$$

$$\Rightarrow \quad f_\mathrm{s} = m\left(2\omega v' + \omega^2 b + \frac{v'^2}{b}\right) \le \mu_\mathrm{s} N = \mu_\mathrm{s} mg \quad \text{(for no slipping)}$$

$$\Rightarrow \quad \frac{v'^2}{b} + 2\omega v' + \omega^2 b - \mu_s g \ \leq \ 0. \tag{2}$$

For maximum v', set $\leq$ to $=$ and solve:

$$v'_{\text{max}} = \frac{-2\omega \pm \sqrt{4\omega^2 - \frac{4}{b}(\omega^2 b - \mu_s g)}}{2/b} = -\omega b + \sqrt{\mu_s b g},$$

with the positive root chosen so that v'_{max} has a chance of being positive. Even then, ω could be large enough so that no positive velocity is consistent with no slipping, in which case the bug would slide off the turntable regardless of its speed.

b) By changing the direction of $\vec{v}'$, all that changes from part a is the sign of the Coriolis force:

$$-2m\vec{\omega} \times \vec{v}' = -2m\omega v' \hat{e}'_r.$$

Thus, (2) becomes:

$$\frac{v'^2}{b} - 2\omega v' + \omega^2 b - \mu_s g \ \leq \ 0 \quad \Rightarrow \quad v'_{\text{max}} = \omega b + \sqrt{\mu_s b g},$$

where the positive root has been chosen for the maximum speed.

So how do we interpret the other root? If $\omega b - \sqrt{\mu_s b g} > 0$, this is actually the *minimum* speed the bug must walk to counter enough of the rotation to prevent slipping. Any slower, and the *net* rotational velocity is too large, causing the bug to slide off the turntable. In this case, any speeds in the range $\omega b - \sqrt{\mu_s b g} < v' < \omega b + \sqrt{\mu_s b g}$ will give the bug a net rotation speed low enough to allow the limited static friction to provide the necessary centripetal force to keep the bug in its circular path.

Problem 4 (FC 5.13) A pebble drops from the observation deck of the CN Tower ($h = 433$ m) at latitude $\lambda = 43.6°$ N. Ignoring air resistance, find the deflection caused by the Coriolis force by the time the pebble reaches the ground.

Solution: With $\vec{r}'_0 = (x'_0, y'_0, z'_0) = (0, 0, h)$, $\vec{v}'_0 = (\dot{x}'_0, \dot{y}'_0, \dot{z}'_0) = (0, 0, 0)$, equations 5.4.15a,b,c (eds. 6 and 7) become:

$$x'(t) = \tfrac{1}{3}\omega g t^3 \cos\lambda; \qquad y'(t) = 0; \qquad z'(t) = h - \tfrac{1}{2} g t^2. \tag{1}$$

When the pebble falls, $z'(t) = 0 \ \Rightarrow \ gt^2 = 2h; \ t = \sqrt{2h/g}$. Thus, the first of equations (1) becomes:

$$x'\big|_{z'=0} = \frac{\omega g}{3}\left(\frac{2h}{g}\right)^{3/2} \cos\lambda = \frac{\omega}{3}\sqrt{\frac{8h^3}{g}} \cos\lambda,$$

as found in example 5.4.1 of the text (eds. 6 and 7). This is the deflection, which is completely in the east (positive) or west (negative) direction (no north-south component).

In this case,

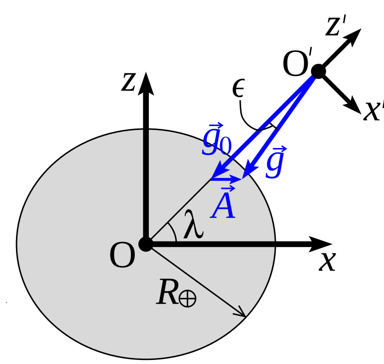
$$x'\Big|_{z'=0} = x'_{\rm def} \sim \frac{7.29\times10^{-5}}{3}\sqrt{\frac{8(433)^3}{9.81}}\cos 43.6^\circ \sim \boxed{0.143 \text{ m to the east.}}$$

Problem 5 In class, we showed that the effective acceleration of gravity, $\vec{g}$, experienced on earth doesn't actually point to the earth's centre because it is the vector sum of the "true" acceleration of gravity, $\vec{g}_0$, that *does* point to the earth's centre and the inertial acceleration $\vec{A} = (R_\oplus\omega^2\cos\lambda)\hat{\imath}$, as shown in the figure. Here, $R_\oplus$ is the radius of the earth, ω is the angular speed of the earth about its axis, and λ is the latitude. Thus, a plumb bob hanging "vertically" deviates from "true vertical" by a small deflection angle, ϵ.

a) Show that the deflection angle is given by,

$$\tan\epsilon = \frac{\frac{1}{2}R_\oplus\omega^2\sin 2\lambda}{g_0 - R_\oplus\omega^2\cos^2\lambda},$$

where $g_0 \sim 9.823\,\mathrm{m\,s^{-2}}$ is the magnitude of the "true" acceleration of gravity.



b) Show that ϵ is a maximum for $\lambda \approx 45^\circ$, and find that maximum value. What is the approximation you had to make to arrive at this value for λ?

Solution: a) If we resolve $\vec{g}$ in the O′ coordinates, then,

$$\tan\epsilon = \left|\frac{g_{x'}}{g_{z'}}\right|,$$

for a positive value for ϵ. Thus, it remains to find the x' and z' components of $\vec{g}$. To this end and from the blue vectors in the figure,

$$\vec{g} = \vec{g}_0 + \vec{A} = -g_0\hat{k}' + (A\cos\lambda)\hat{k}' + (A\sin\lambda)\hat{\imath}'$$

$$\Rightarrow \quad \frac{g_{x'}}{g_{z'}} = \frac{A\sin\lambda}{-g_0 + A\cos\lambda} = \frac{R_\oplus\omega^2\cos\lambda\sin\lambda}{-g_0 + R_\oplus\omega^2\cos^2\lambda} = \frac{\frac{1}{2}R_\oplus\omega^2\sin 2\lambda}{-g_0 + R_\oplus\omega^2\cos^2\lambda}$$

$$\Rightarrow \quad \boxed{\tan\epsilon = \frac{\frac{1}{2}R_\oplus\omega^2\sin 2\lambda}{g_0 - R_\oplus\omega^2\cos^2\lambda},} \tag{1}$$

as desired, taking the absolute value (since $g_0 > R_\oplus\omega^2$).

An alternative way is through the dot product, which includes $\cos\epsilon$ as part of its definition.

Once again, resolving $\vec{g}_0$ and $\vec{g}$ in the O$'$ coordinates, we have:

$$\vec{g}_0 \;=\; -g_0\hat{k}'; \qquad \text{and} \qquad \vec{g} \;=\; g_{x'}\hat{\imath}' + g_{z'}\hat{k}'.$$

Then, from the definition of the dot product,

$$\vec{g}_0\cdot\vec{g} \;=\; -g_0 g_{z'} \;=\; g_0\sqrt{g_{x'}^2 + g_{z'}^2}\cos\epsilon \quad\Rightarrow\quad \cos\epsilon \;=\; \frac{-g_{z'}}{\sqrt{g_{x'}^2 + g_{z'}^2}}$$

$$\Rightarrow \quad \tan\epsilon \;=\; \frac{\sin\epsilon}{\cos\epsilon} \;=\; \frac{\sqrt{1-\cos^2\epsilon}}{\cos\epsilon} \;=\; \sqrt{1 - \frac{g_{z'}^2}{g_{x'}^2 + g_{z'}^2}}\;\frac{\sqrt{g_{x'}^2 + g_{z'}^2}}{-g_{z'}}$$

$$= \sqrt{\frac{g_{x'}^2 + \cancel{g_{z'}^2} - \cancel{g_{z'}^2}}{\cancel{g_{x'}^2 + g_{z'}^2}}}\;\frac{\cancel{\sqrt{g_{x'}^2 + g_{z'}^2}}}{-g_{z'}} \;=\; -\frac{g_{x'}}{g_{z'}}.$$

which is essentially where we started in the first method. Substituting,

$$g_{x'} \;=\; A\sin\lambda \qquad \text{and} \qquad g_{z'} \;=\; -g_0 + A\cos\lambda,$$

we get,

$$\tan\epsilon \;=\; \frac{A\sin\lambda}{g_0 - A\cos\lambda} \;=\; \frac{R_\oplus\omega^2\cos\lambda\sin\lambda}{g_0 - R_\oplus\omega^2\cos^2\lambda} \;=\; \frac{\frac{1}{2}R_\oplus\omega^2\sin 2\lambda}{g_0 - R_\oplus\omega^2\cos^2\lambda},$$

as before.

The additional awkwardness in the second method is because a cosine has to be converted to a tangent; the first method started right off with a tangent. Note that to retains one's sanity, one should delay substituting in for $g_{x'}$, $g_{z'}$, and A until as late as possible.

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b) Maximising $\tan\epsilon$ also maximises ϵ, and so to find the latitude at which the deviation is the greatest, we differentiate equation (1) with respect to λ and set $d(\tan\epsilon)/d\lambda = 0$. Thus,

$$\frac{d\tan\epsilon}{d\lambda} \;=\; \frac{R_\oplus\omega^2\cos 2\lambda\big(g_0 - R_\oplus\omega^2\cos^2\lambda\big) - 2R_\oplus\omega^2\cos\lambda\sin\lambda\big(\frac{1}{2}R_\oplus\omega^2\sin 2\lambda\big)}{\big(g_0 - R_\oplus\omega^2\cos^2\lambda\big)^2} \;=\; 0$$

$$\Rightarrow \quad \big(2\cos^2\lambda - 1\big)\big(\alpha - \cos^2\lambda\big) \;=\; 2\cos^2\lambda\sin^2\lambda \;=\; 2\cos^2\lambda\big(1-\cos^2\lambda\big),$$

where $\alpha \equiv \dfrac{g_0}{R_\oplus\omega^2}$ to ease up on the writing. Continuing,

$$2\alpha\cos^2\lambda - \cancel{2\cos^4\lambda} - \alpha + \cos^2\lambda \;=\; 2\cos^2\lambda - \cancel{2\cos^4\lambda} \Rightarrow \quad \alpha\big(2\cos^2\lambda - 1\big) \;=\; \cos^2\lambda$$

$$\Rightarrow \quad 2\cos^2\lambda - 1 \;=\; \frac{\cos^2\lambda}{\alpha} \;\sim\; \frac{\cos^2\lambda}{290.0},$$

using known values $R_\oplus \sim 6.371\times 10^6\,\mathrm{m}$, $\omega \sim 7.292\times 10^{-5}\,\mathrm{rad\,s^{-1}}$, and $g_0 = 9.823\,\mathrm{m\,s^{-2}}$.

Thus, taking $\underline{\underline{\alpha \gg 1}}$, we set the RHS to approximately zero and get,

$$2\cos^2\!\lambda - 1 \;\approx\; 0 \quad\Rightarrow\quad \cos^2\!\lambda \;\approx\; \frac{1}{2} \quad\Rightarrow\quad \underline{\underline{\lambda \;\approx\; 45^\circ}},$$

is the latitude for the greatest deviation from true vertical, as desired[1]. Substituting $\lambda = 45^\circ$ into equation (1), we get,

$$\tan\epsilon_{\rm max} \;\approx\; \epsilon_{\rm max} \;\approx\; \frac{\frac{1}{2}R_\oplus\omega^2}{g_0 - \frac{1}{2}R_\oplus\omega^2} \;=\; \frac{1}{2\alpha - 1} \;\sim\; \underline{\underline{0.00173\,\text{rad}}},$$

or about 0.10°.

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[1]By not making the approximation, $\lambda = 44.95^\circ$, which is about 5 km north of the Halifax airport!