# First Practise Midterm Solutions

PHYS 2302 (Mechanics I); D. A. Clarke

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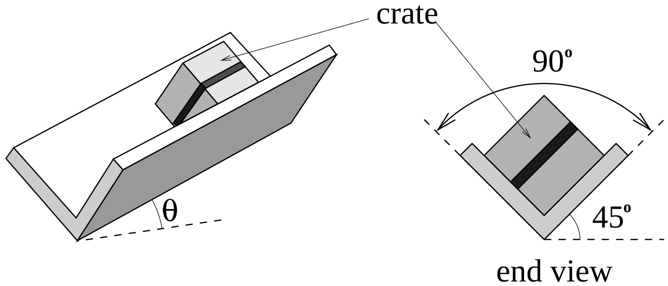
**Problem 1. Crate sliding down a V-ramp:** A crate slides down a right-angle trough inclined to the horizontal at an angle $\theta$ as shown in the figure. If the coefficient of kinetic friction between the sides of the crate and the surface of the trough is $\mu_k$, find the acceleration of the crate.

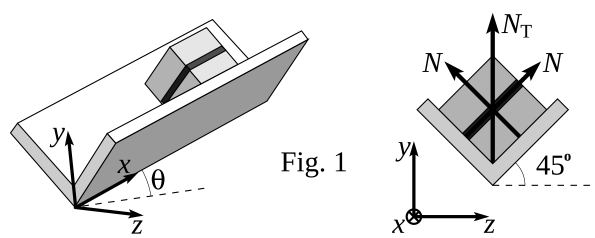


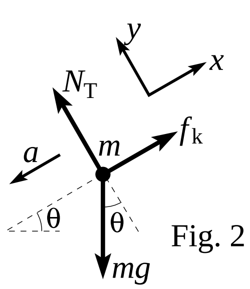

*Hint*: Normal forces are exerted on the crate by both sides of the trough.

*Solution*: From Fig. 1 to the right, we see that the $z$-components of the normal forces cancel, leaving a total normal force $\vec{N}_T = \sqrt{2}N\hat{\jmath}$. This simplification allows us to construct our free body diagram (Fig. 2 below) entirely in the $x$-$y$ plane.

On the other hand, the kinetic frictional force responds to the normal forces from each surface directly, and $\vec{f}_k = \mu_k(2N)\hat{\imath}$.

Thus, from the FBD, Newton's 2nd Law for this problem is written:




$$x/ \quad f_k - mg\sin\theta = 2\mu_k N - mg\sin\theta = -ma; \tag{1}$$

$$y/ \quad N_T - mg\cos\theta = \sqrt{2}N - mg\cos\theta = 0 \tag{2}$$

From equation (2), $N = \dfrac{mg}{\sqrt{2}}\cos\theta$. Substituting this into equation (1), we get:

$$2\mu_k\frac{\cancel{m}g}{\sqrt{2}}\cos\theta - \cancel{m}g\sin\theta = -\cancel{m}a \quad \Rightarrow \quad \boxed{a = g(\sin\theta - \sqrt{2}\mu_k\cos\theta).}$$

Short and sweet.

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**Problem 2. Sliding block:** A metal block slides along a horizontal surface lubricated with a heavy oil that imposes on the block a resistive force given by:

$$F(v) = -cv^{3/2}, \tag{3}$$

where $v$ is the speed of the block, $c$ is a constant, and the negative sign indicates this force causes the block to slow down.

a) (12 points) If at $x = 0$ the speed of the block is $v_0$, show that the speed of the block after it slides a distance $x$ is given by,

$$v(x) = \left(\sqrt{v_0} - \frac{cx}{2m}\right)^2.$$

b) (2 points) Find $X$, the maximum distance travelled by the block.

c) (6 points) By solving the differential equation,

$$F(v) = m\frac{dv}{dt},$$

determine whether the block reaches $X$ asymptotically (*i.e.*, as $t \to \infty$) or after a finite time, $T$.

*Solution*: a) To find velocity as a function of distance, $v(x)$, we need Newton's second law in terms of $v$ and $x$. Thus,

$$F = ma = m\frac{dv}{dt} = m\frac{dx}{dt}\frac{dv}{dx} = mv\frac{dv}{dx} = -cv^{3/2},$$

which is a separable first order ODE in $v$ and $x$. Thus,

$$v^{-1/2}dv = -\frac{c}{m}\,dx \quad \Rightarrow \quad \int_{v_0}^{v}(v')^{-1/2}dv' = -\frac{c}{m}\int_0^x dx',$$

where integration is taken from $x = 0$ where the speed of the block is $v_0$ to a distance $x$ where the speed is $v$. Doing the integrals, we get:

$$2\sqrt{v'}\Big|_{v_0}^{v} = -\frac{c}{m}\,x'\Big|_0^x \quad \Rightarrow \quad 2\left(\sqrt{v} - \sqrt{v_0}\right) = -\frac{cx}{m}$$

$$\Rightarrow \quad \boxed{v(x) = \left(\sqrt{v_0} - \frac{cx}{2m}\right)^2,} \tag{4}$$

as desired.

b) To find the maximum distance, $X$, the block slides, set $v = 0$ in equation (4) to get:

$$0 = \left(\sqrt{v_0} - \frac{cX}{2m}\right)^2 \quad \Rightarrow \quad \boxed{X = \frac{2m\sqrt{v_0}}{c}.}$$

c) To find the time it takes for the block to stop, we start with Newton's second law in terms of $v$ and $t$:

$$F = m\frac{dv}{dt} = -cv^{3/2} \quad \Rightarrow \quad v^{-3/2}dv = -\frac{c}{m}\,dt \quad \Rightarrow \quad \int_{v_0}^{0} v^{-3/2}dv = -\frac{c}{m}\int_0^T dt$$

$$\Rightarrow \quad -\frac{2}{\sqrt{v}}\bigg|_{v_0}^{0} = -\frac{c}{m}\,t\bigg|_0^T \quad \Rightarrow \quad T = \frac{2m}{c}\,\frac{1}{\sqrt{v}}\bigg|_{v_0}^{0} \to \infty.$$

Thus, the block reaches its maximum distance asymptotically. This points to a weakness in the model for the viscosity of the oil, equation (3), since we know perfectly well that the block comes to a dead stop well within our lifetimes!

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**Problem 3. SHO plug and chug:** Consider the position of a simple harmonic oscillator as a function of time given by,

$$x(t) = C\cos(\omega_0 t - \phi_0), \tag{5}$$

where $C$ is the amplitude of oscillation, $\omega_0$ is the angular frequency of oscillation, and $\phi_0$ is the phase of oscillation determining where the oscillator is at $t = 0$.

a) (5 points) Show by direct substitution that equation (5) solves the differential equation of motion for a simple harmonic oscillator, as given on the formula sheets.

b) (15 points) If, at $t = 0$, $x(0) = x_0$ and $v(0) = \dot{x}(0) = v_0$, find $C$ and $\phi_0$ in terms of quantities given.

*Solution*: a) As given on the formula sheets, the differential equation of motion for an SHO is:

$$\frac{d^2x}{dt^2} = -\omega_0^2 x. \tag{6}$$

Substituting equation (5) into the LHS of equation (6), we get:

$$\frac{d^2x}{dt^2} = -\omega_0^2 C\cos(\omega_0 t - \phi_0) = -\omega_0^2 x,$$

as desired.

b) For this part, we need the velocity of the oscillator given by the time-derivative of equation (5):

$$v(t) = \frac{dx}{dt} = -C\omega_0\sin(\omega_0 t - \phi_0). \tag{7}$$

Then, applying initial conditions, we get:

$$x(0) = C\cos(-\phi_0) = C\cos\phi_0 = x_0; \tag{8}$$

$$v(0) \;=\; -C\omega_0 \sin(-\phi_0) \;=\; v_0 \quad \Rightarrow \quad C \sin\phi_0 \;=\; \frac{v_0}{\omega_0}. \tag{9}$$

Dividing equation (9) by equation (8), we get:

$$\frac{\not{C} \sin\phi_0}{\not{C} \cos\phi_0} \;=\; \frac{v_0/\omega_0}{x_0} \quad \Rightarrow \quad \boxed{\tan\phi_0 \;=\; \frac{v_0}{\omega_0 x_0}.}$$

Next, adding the squares of equations (8) and (9), we get:

$$C^2 \cos^2\!\phi_0 + C^2 \sin^2\!\phi_0 \;=\; C^2 \;=\; x_0^2 + \frac{v_0^2}{\omega_0^2} \quad \Rightarrow \quad \boxed{C \;=\; \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}.}$$

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