# Second Practise Midterm Solutions

PHYS 2302 (Mechanics I); D. A. Clarke

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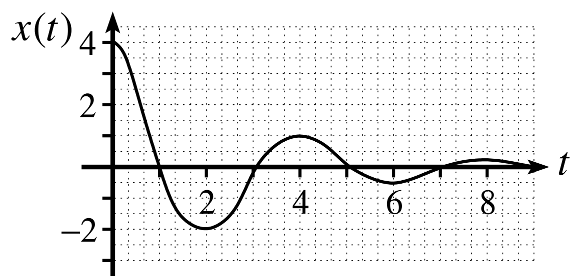
**Problem 1. Damped harmonic oscillator:** The plot shows the position of an under-damped harmonic oscillator as a function of time. The units on the axes are arbitrary but, if it helps, you can think of the units on the ordinate as cm and those on the abscissa as s.

As given on the formula sheets, the equation of motion for such an oscillator is,

$$x(t) \;=\; x_0 e^{-\gamma t}\frac{\omega_0}{\omega_\mathrm{d}}\cos(\omega_\mathrm{d} t - \theta_0), \qquad (1)$$

where all quantities are defined on the formula sheets.




a) (11 points; 3 for $\gamma$, 2 for each of the others) By reading off whatever data you may need from the plot, find numerical values good to three significant figures for each of the five constants in equation (1).

*Caution*: Read off only values that can be read accurately (such as the period, $T$). Values that cannot be read accurately from the plot (such as the phase, $\theta_0$) should be calculated from appropriate expressions using values that can be read accurately.

b) (9 points) Using your quantities in part a, find values good to three significant figures for each of:

*i*) the resonant frequency, $\omega_\mathrm{r}$;

*ii*) the quality factor, $Q_\mathrm{d}$;

*iii*) the resonant amplitude, $A_\mathrm{max}$, for $F_0/m = 10.0$ in the arbitrary units of the plot.

*Caution*: In order to ensure accuracy to three significant figures, your input values from part a will have to be good to 4 or even 5 significant figures to avoid accumulating round-off error in the third significant figure reported.

*Solution*: a) At $t = 0$,

$$x(0) \;=\; x_0\frac{\omega_0}{\omega_\mathrm{d}}\cos\theta_0 \;=\; x_0\frac{\omega_0}{\omega_\mathrm{d}}\frac{\omega_\mathrm{d}}{\omega_0} \;=\; x_0,$$

since from the formula sheets,

$$\sin\theta_0 \;=\; \frac{\gamma}{\omega_0} \quad\Rightarrow\quad \cos\theta_0 \;=\; \sqrt{1-\frac{\gamma^2}{\omega_0^2}} \;=\; \frac{\sqrt{\omega_0^2-\gamma^2}}{\omega_0} \;=\; \frac{\omega_\mathrm{d}}{\omega_0}.$$

Thus,

$$\underline{\underline{x_0 \;=\; x(0) \;=\; 4.00}},$$

as read from the plot.

We can also read the period from the plot,

$$T \;=\; 4.00 \quad\Rightarrow\quad \omega_\mathrm{d} \;=\; \frac{2\pi}{T} \;=\; \frac{\pi}{2} \quad\Rightarrow\quad \underline{\underline{\omega_\mathrm{d} \;\sim\; 1.5708}}.$$

For the damping coefficient, $\gamma$, let the amplitude after one period be $x_T$. Then,

$$x_T \;=\; x(T) \;=\; x_0 e^{-\gamma T}\frac{\omega_0}{\omega_\mathrm{d}}\cos(\underbrace{\omega_\mathrm{d} T}_{2\pi} - \theta_0) \;=\; x_0 e^{-\gamma T}\frac{\omega_0}{\omega_\mathrm{d}}\cos(\theta_0) \;=\; x_0 e^{-\gamma T}$$

$$\Rightarrow\quad e^{-\gamma T} \;=\; \frac{x_T}{x_0} \quad\Rightarrow\quad -\gamma T \;=\; \ln\left(\frac{x_T}{x_0}\right) \quad\Rightarrow\quad \gamma \;=\; \frac{1}{T}\ln\left(\frac{x_0}{x_T}\right).$$

Noting from the plot that $x_0 = 4.00$, $x_T = 1.00$, and $T = 4.00$, we have,

$$\underline{\underline{\gamma \;=\; \frac{\ln 4.00}{4.00} \;\sim\; 0.34657.}}$$

Next, from the formula sheets,

$$\omega_0 \;=\; \sqrt{\omega_\mathrm{d}^2 + \gamma^2} \quad\Rightarrow\quad \underline{\underline{\omega_0 \;\sim\; 1.6086}},$$

using the values for $\gamma$ and $\omega_\mathrm{d}$ already calculated.

Finally, using the values for $\omega_0$ and $\omega_\mathrm{d}$ already calculated,

$$\cos\theta_0 \;=\; \frac{\omega_\mathrm{d}}{\omega_0} \;\sim\; 0.97651 \quad\Rightarrow\quad \underline{\underline{\theta_0 \;\sim\; 0.21716\,\mathrm{rad} \;\sim\; 12.442^\circ}}.$$

b) *i*) From the formula sheets, the resonant frequency is given by,

$$\omega_\mathrm{r} \;=\; \sqrt{\omega_0^2 - 2\gamma^2} \;=\; \sqrt{\omega_\mathrm{d}^2 - \gamma^2} \quad\Rightarrow\quad \underline{\underline{\omega_\mathrm{r} \;\sim\; 1.5321}}.$$

*ii*) From the formula sheets, the quality factor is given by:

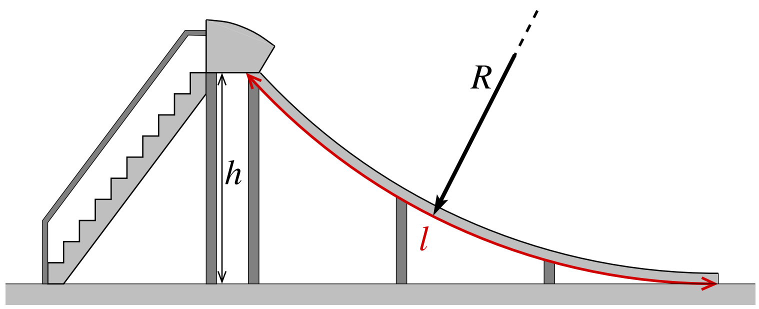
$$Q_\mathrm{d} \;=\; \frac{\omega_\mathrm{d}}{2\gamma} \;=\; \frac{\pi}{4\gamma} \quad\Rightarrow\quad \underline{\underline{Q_\mathrm{d} \;\sim\; 2.2662}}.$$

*iii*) Finally, from the formula sheets, the resonant amplitude is given by:

$$A_\mathrm{max} \;=\; \frac{F_0/m}{2\gamma\omega_\mathrm{d}} \;=\; \frac{10.0}{\pi\gamma} \quad\Rightarrow\quad \underline{\underline{A_\mathrm{max} \;\sim\; 9.1845}}.$$

Presuming the units suggested by the problem, these would be cm.

**Problem 2. A playground slide:** The figure depicts a playground slide in the form of an arc of a circle with a height $h = 4.00\,\text{m}$, radius of curvature $R = 12.0\,\text{m}$, and with the ground tangent to the circular arc. A 25.0 kg child starts from rest at the top of the slide and has a speed of $6.20\,\text{m}\,\text{s}^{-1}$ at the bottom.




a) Show that the arc-length of the slide is given by: $l = R\cos^{-1}\dfrac{R-h}{R}$.

b) Use the work-kinetic theorem to find the average friction force acting on the child by the slide, where the average value of a quantity, $q$, over a path P of length $l$ is,

$$\langle q\rangle_{\text{P}} \equiv \frac{1}{l}\int_{\text{P}} q\,dl. \qquad (1)$$

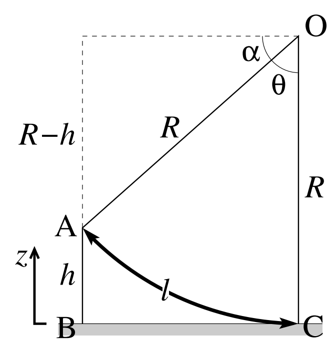
c) What is the maximum normal force exerted by the slide on the child?

Give numerical values for each answer accurate to three significant figures.

*Solution*: a) Finding the arc-length of the slide is a geometry problem. As shown in the figure, the arc-length of the slide, $l$, subtends an angle $\theta$ from the centre of curvature, O. The compliment of $\theta$ is $\alpha$, where, as is evident from the figure,

$$\sin\alpha = \frac{R-h}{R} = \cos\theta$$

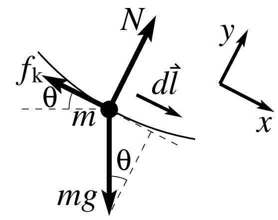
$$\Rightarrow \boxed{l = R\theta = R\cos^{-1}\frac{R-h}{R} \sim 10.1\,\text{m},} \qquad (2)$$




using the values given in the problem.

b) To use the W-K theorem, we need expressions for the work done by all forces acting on the child, all shown in the FBD.

Since gravity is a conservative force, the work it does is path-independent and instead of the curved path AC in the figure, we can use path ABC along which gravity does work along AB only:




$$W_{mg} = \int_h^0 m\vec{g}\cdot d\vec{z} = -mg\int_h^0 dz = mgh, \qquad (3)$$

where $z$ is the vertical coordinate.

As usual, the normal force does no work, leaving us with the work done by friction:

$$W_{f_k} = \int_A^C \vec{f}_k \cdot d\vec{l} = -\int_A^C f_k\,dl = -\langle f_k\rangle l, \qquad (4)$$

using equation (1) for the average frictional force, where $d\vec{l}$ is an infinitesimal displacement along the slide, and where $l$ is given by equation (2).

Thus, from the work-kinetic theorem:

$$\sum W = W_{mg} + \cancelto{0}{W_N} + W_{f_k} = mgh - \langle f_k\rangle l = K - K_0 = \frac{1}{2}mv_C^2,$$

where $v_C$ is the child's speed at C, and $K_0 = 0$ since the child starts at A from rest. Thus,

$$\boxed{\langle f_k\rangle = \frac{m}{l}\left(gh - \tfrac{1}{2}v_C^2\right) \sim 49.6\,\mathrm{N},}$$

where $l$ is given by equation (2) and using the values given in the problem.

c) Finally, from the FBD, we have:

$$x/\;\; mg\sin\theta - f_k = ma_x; \qquad y/\;\; N - mg\cos\theta = ma_y.$$

Only the $y$-component is relevant here. For circular motion, $a_y$ must be the centripetal acceleration, and thus:

$$N = mg\cos\theta + m\frac{v^2}{R},$$

which is a maximum at the bottom of the slide where both $\cos\theta$ and $v$ attain their maximum values. Thus,

$$\boxed{N_{\max} = mg\left(1 + \frac{v_C^2}{Rg}\right) \sim 1.33\,mg \sim 325.\,\mathrm{N}.}$$

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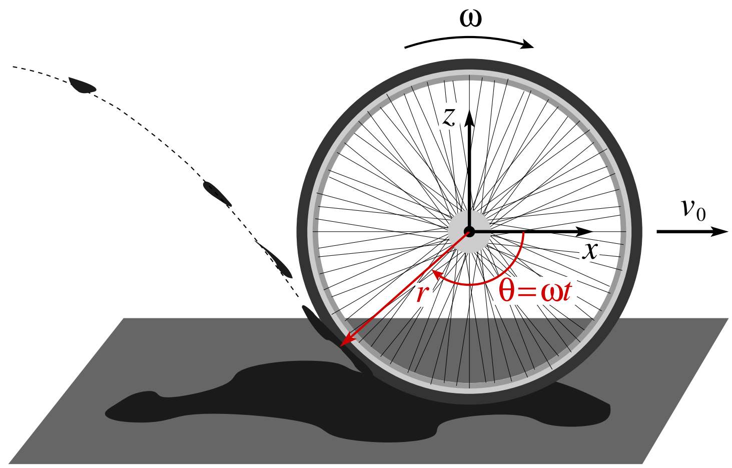
**Problem 3. Mud flung from a tire:**

a) (16 points) As shown in the figure, blobs of mud are thrown from the rim of a rolling bicycle wheel of radius $r$. If the forward speed of the wheel axle is $v_0$ and constant, show that the greatest height *above the ground* that the mud can attain is given by,

$$h_{\max} = r + \frac{v_0^2}{2g} + \frac{r^2 g}{2v_0^2}.$$

*Hint*: This is a projectile problem, where the height attained by a blob of mud (the projectile) needs to be maximised with respect to the launching angle, $\theta$.



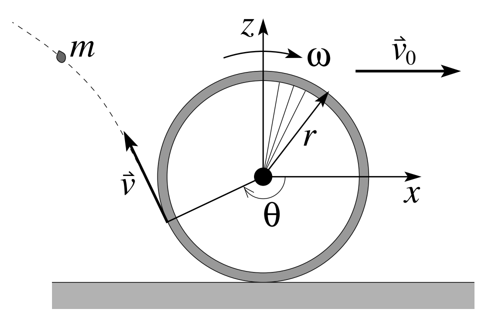

b) (4 points) If $r = 0.36$ m and $v_0 = 6$ m s$^{-1}$, at what value of $\theta$ does the mud attaining the greatest height leave the wheel?

*Solution*: a) Relative to the non-rotating $x$-$z$ coordinate system fixed to the axle, the velocity of any point on the rim of the wheel is:

$$\vec{v}_{\rm rim} = v_0(-\hat{\imath}\sin\theta - \hat{k}\cos\theta), \qquad (2)$$

where $v_0 = \omega r$, and where $\theta$, as shown in the figure, is taken to be a positive quantity (opposite to normal convention). The position of this point on the rim is evidently:

$$\vec{r}_{\rm rim} = r(\hat{\imath}\cos\theta - \hat{k}\sin\theta). \qquad (3)$$




Once launched, the height of the blob of mud and its $z$-velocity are given by:

$$z - z_0 = v_{0z}t - \tfrac{1}{2}gt^2 \quad \Rightarrow \quad v_z = \dot{z} = v_{0z} - gt. \qquad (4)$$

At its highest point, $z = h$, $v_z = 0$, and the second of equations (4) $\Rightarrow$

$$t_{\rm h} = \frac{v_{0z}}{g}. \qquad (5)$$

Substituting equation (5) into the first of equations (4) yields:

$$h = z_0 + \frac{v_{0z}^2}{2g}. \tag{6}$$

Now, from equation (2) we have $v_{0z} = -v_0 \cos\theta$ and from equation (3), $z_0 = -r\sin\theta$. Thus, equation (6) becomes:

$$h = -r\sin\theta + \frac{v_0^2 \cos^2\theta}{2g}. \tag{7}$$

To find the maximum height, set:

$$\frac{dh}{d\theta} = -r\cos\theta - \frac{v_0^2}{g}\cos\theta\sin\theta = 0$$

$$\Rightarrow \quad \sin\theta = -\frac{rg}{v_0^2} \qquad \text{and} \qquad \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{r^2 g^2}{v_0^4}. \tag{8}$$

Substituting equations (8) into equation (7) gives us:

$$h_{\text{max}} = \frac{r^2 g}{v_0^2} + \frac{v_0^2}{2g}\left(1 - \frac{r^2 g^2}{v_0^4}\right) = \frac{v_0^2}{2g} + \frac{r^2 g}{2v_0^2}.$$

This is all relative to the axle of the wheel. Relative to the ground, we must add $r$ to get:

$$\boxed{h_{\text{max}} = r + \frac{v_0^2}{2g} + \frac{r^2 g}{2v_0^2},}$$

as desired.

b) If $r = 0.36$ m and $v_0 = 6\ \text{m}\,\text{s}^{-1}$, the first of equations (8) gives us:

$$\sin\theta = -\frac{0.36}{36} 9.81 = -0.0981$$

$$\Rightarrow \quad \boxed{\theta = 185.6^\circ,}$$

slightly higher than horizontal.