Solutions to Extra Problems

PHYS 2302 (Mechanics I); D. A. Clarke

Problem 1.

Solution: Starting with the hint, since $\vec{D} = \vec{A} \times (\vec{B} \times \vec{C}) \perp \vec{B} \times \vec{C}$, then $\vec{D}$ must lie in the plane spanned by $\vec{B}$ and $\vec{C}$, and we may write:

$$\vec{D} \;=\; \beta\vec{B} + \gamma\vec{C}. \tag{1}$$

Now, $\vec{D} \perp \vec{A}$ as well, and thus,

$$\vec{A}\cdot\vec{D} \;=\; \beta(\vec{A}\cdot\vec{B}) + \gamma(\vec{A}\cdot\vec{C}) \;=\; 0 \quad\Rightarrow\quad \beta \;=\; -\frac{\gamma}{(\vec{A}\cdot\vec{B})}\,(\vec{A}\cdot\vec{C}) \;\equiv\; \alpha(\vec{A}\cdot\vec{C}), \tag{2}$$

where α is a scalar. Further,

$$\gamma(\vec{A}\cdot\vec{C}) \;=\; -\beta(\vec{A}\cdot\vec{B}) \;=\; -\alpha(\vec{A}\cdot\vec{C})(\vec{A}\cdot\vec{B}) \quad\Rightarrow\quad \gamma \;=\; -\alpha(\vec{A}\cdot\vec{B}). \tag{3}$$

Substituting equations (2) and (3) into equation (1) gives us:

$$\vec{D} \;=\; \alpha(\vec{A}\cdot\vec{C})\vec{B} - \alpha(\vec{A}\cdot\vec{B})\vec{C} \;=\; \alpha\big[\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\big]. \tag{4}$$

To prove the identity, then, it remains to show that $\alpha = 1$. To this end, we observe that equation (4) is true for *any* set of vectors in *any* coordinate system with the only proviso that $\vec{B} \neq \vec{C}$ so that $\vec{B}$ and $\vec{C}$ uniquely define a plane. Thus, without loss of generality[1], suppose $\vec{A} = \vec{C} = \hat{\imath}$ and $\vec{B} = \hat{\jmath}$. Then equation (4) becomes:

$$\begin{aligned}
\vec{D} \;&=\; \vec{A}\times(\vec{B}\times\vec{C}) \;=\; \hat{\imath}\times(\hat{\jmath}\times\hat{\imath}) \;=\; \hat{\imath}\times(-\hat{k}) \;=\; \hat{\jmath} \\
&=\; \alpha\big[(\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}\big] \;=\; \alpha\big[\cancelto{1}{(\hat{\imath}\cdot\hat{\imath})}\hat{\jmath} - \cancelto{0}{(\hat{\imath}\cdot\hat{\jmath})}\hat{\imath}\big] \;=\; \alpha\hat{\jmath},
\end{aligned}$$

proving that $\alpha = 1$ and thus the assertion.

Problem 2.

Solution: Start with:

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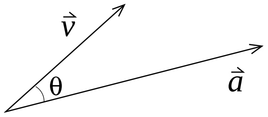
[1] In practise, what it means to be able to choose any vectors from any coordinate system is that no matter what vectors we choose to evaluate α, we'll get the same result. Don't believe it? Try it!

$$\vec{v}\cdot\vec{a} \;=\; va\cos\theta \quad\Rightarrow\quad \cos\theta \;=\; \frac{\vec{v}\cdot\vec{a}}{va}. \tag{1}$$



In polar coordinates (using equations 1.11.7 and 1.11.9),

$$\vec{v} \;=\; bke^{kt}\hat{e}_r + be^{kt}c\hat{e}_\theta \;=\; be^{kt}(k\hat{e}_r + c\hat{e}_\theta)$$

$$\vec{a} \;=\; (bk^2e^{kt} - be^{kt}c^2)\hat{e}_r + 2bke^{kt}c\hat{e}_\theta \;=\; be^{kt}\big((k^2-c^2)\hat{e}_r + 2kc\hat{e}_\theta\big)$$

$$\Rightarrow\quad \vec{v}\cdot\vec{a} \;=\; b^2e^{2kt}\big(k(k^2-c^2)+2kc^2\big) \;=\; b^2e^{2kt}k(k^2+c^2). \tag{2}$$

Further,

$$v \;=\; |\vec{v}| \;=\; be^{kt}\sqrt{k^2+c^2}, \tag{3}$$

and

$$a \;=\; be^{kt}\sqrt{(k^2-c^2)^2+4k^2c^2} \;=\; be^{kt}\sqrt{k^4+c^4+2k^2c^2} \;=\; be^{kt}(k^2+c^2). \tag{4}$$

Substituting equations (2)–(4) into equation (1), we get:

$$\cos\theta \;=\; \frac{b^2e^{2kt}k(k^2+c^2)}{b^2e^{2kt}(k^2+c^2)^{3/2}} \;=\; \frac{k}{\sqrt{k^2+c^2}} \;=\; \text{constant}.$$

Problem 3.

Solution: a) Taking the derivative of $\vec{r}\cdot(\vec{v}\times\vec{a})$, we get:

$$\frac{d}{dt}\big(\vec{r}\cdot(\vec{v}\times\vec{a})\big) \;=\; \frac{d\vec{r}}{dt}\cdot(\vec{v}\times\vec{a}) + \vec{r}\cdot\frac{d}{dt}(\vec{v}\times\vec{a})$$

$$= \cancelto{0}{\vec{v}\cdot\underbrace{(\vec{v}\times\vec{a})}_{\perp\,\vec{v}}} + \vec{r}\cdot\left(\underbrace{\frac{d\vec{v}}{dt}\times\vec{a}}_{\vec{a}\times\vec{a}\,=\,0} + \vec{v}\times\frac{d\vec{a}}{dt}\right) \;=\; \vec{r}\cdot\left(\vec{v}\times\frac{d\vec{a}}{dt}\right),$$

as desired.

b) Consider $v^2 = \vec{v}\cdot\vec{v}$, and take its time derivative:

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) \;=\; \frac{d}{dt}v^2 \;=\; 2v\dot{v}$$

$$= \vec{v}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt}\cdot\vec{v} \;=\; \vec{v}\cdot\vec{a} + \vec{a}\cdot\vec{v} \;=\; 2\vec{v}\cdot\vec{a},$$

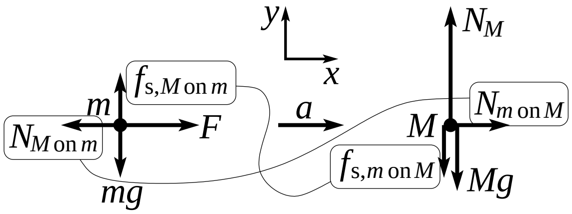
and thus $\vec{v}\cdot\vec{a} = v\dot{v}$, as desired. It follows that if $|\vec{v}|$ is constant, $\dot{v} = 0$ and $\vec{v}\perp\vec{a}$.

Note that $a = |\vec{a}\,| \neq \dot{v}$ necessarily, since the latter is a measure of the acceleration along the direction of motion only. Thus, for example, if the particle maintains a constant speed, v, around a circle of radius r, $\dot{v} = 0$ but $a = v^2/r \neq 0$ (centripetal acceleration).

Problem 4.

Solution: Since the critical information (that m is on the verge of slipping along M) has to do with the interface between m and M, we must consider the two masses as separate free bodies and assess the forces on each. Thus, consider the free-body diagrams for m and M:

$$\begin{aligned}
m\colon\quad x/ &\qquad -N_{M\,\mathrm{on}\,m} + F = ma; &(1)\\
y/ &\qquad f_{\mathrm{s},M\,\mathrm{on}\,m} - mg = 0, &(2)\\
M\colon\quad x/ &\qquad N_{m\,\mathrm{on}\,M} = Ma; &(3)\\
y/ &\qquad N_M - f_{\mathrm{s},m\,\mathrm{on}\,M} - Mg = 0. &(4)
\end{aligned}$$



Note that F acts on m only, and thus does not appear on the FBD for M. The fact that m is being pushed by F is communicated to M via the normal force m exerts on M, $N_{m\,\mathrm{on}\,M}$.

Note further there are two "action-reaction pairs": $N_{m\,\mathrm{on}\,M} = N_{M\,\mathrm{on}\,m} = N$ and $f_{\mathrm{s},m\,\mathrm{on}\,M} = f_{\mathrm{s},M\,\mathrm{on}\,m} = f_\mathrm{s}$. Of these, students will often miss $f_{\mathrm{s},m\,\mathrm{on}\,M}$ on the FBD for M. Particularly when assessing the FBDs of two or more interacting objects, use Newton's third law to make sure you haven't missed any "reaction forces".

Finally note that since m is on the verge of slipping, $f_\mathrm{s} = \mu_\mathrm{s} N$. If m were not on the verge of slipping, the best we could say is $f_\mathrm{s} < \mu_\mathrm{s} N$.

To find a and F, note that equation (4) brings in a fourth unknown, N_M, which we do not care about. Thus, we need only consider equations (1)–(3). Starting from equation (2),

$$f_\mathrm{s} = mg = \mu_\mathrm{s} N \quad\Rightarrow\quad N = \frac{mg}{\mu_\mathrm{s}}.$$

Substituting this into each of equations (1) and (3), we get:

$$F - \frac{mg}{\mu_\mathrm{s}} = ma; \tag{5}$$

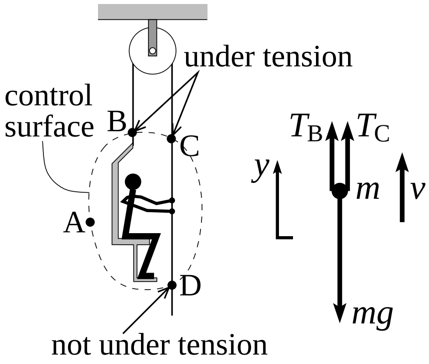
$$\frac{mg}{\mu_\mathrm{s}} = Ma \quad\Rightarrow\quad \boxed{a = \frac{mg}{\mu_\mathrm{s} M}.} \tag{6}$$

Substitute equation (6) into (5) to get minimum force:

$$F - \frac{mg}{\mu_\mathrm{s}} = m\frac{mg}{\mu_\mathrm{s} M} \quad\Rightarrow\quad \boxed{F = \frac{mg}{\mu_\mathrm{s}}\left(1 + \frac{m}{M}\right).}$$

Problem 5.

Solution: a) Using the "control surface" described in the hint, start at point A, and move clockwise. At B, a rope *under tension* is encountered exerting an external force on m. If we were to cut the rope at the control surface, the end of the rope left inside would fall down (along with the chair!). Thus, the tension in the rope was acting upward to prevent this from happening, and T_B is added to the FBD pointing *up*.



Similarly, at C we find another rope under tension exerting an upward force, and T_C is added to the FBD pointing up.

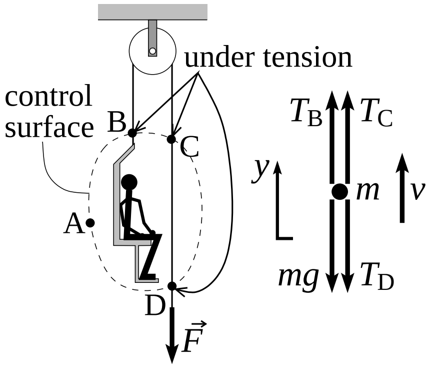
At D, another rope is encountered, but this is *not* under tension, and thus no T_D is added to the FBD.

Including mg pointing downward completes the FBD.

Now, for an ideal pulley and a massless cord, the tension along the cord is uniform $\Rightarrow T_\text{B} = T_\text{C} = T$. Thus, from the FBD and Newton's 2nd law:

$$2T - mg \,=\, 0 \quad \text{(since } v \text{ is constant, } a = 0\text{)} \quad \Rightarrow \quad \boxed{T \,=\, \frac{mg}{2}.}$$

T is provided by the person in the chair pulling on the rope, and thus the person must pull with half the weight of m.



b) If the rope is pulled by someone from below, then each rope crossing the control surface is under tension, including D. Points B and C are as before (cutting the rope at each point causes the end left inside the control surface to fall, thus tension was acting up). However, cutting the rope at D causes the end of the rope left inside the control surface to accelerate *upward* and here, the tension must have been acting *downward*. Thus, add T_D downward to the FBD.

Once again, $T_\text{B} = T_\text{C} = T_\text{D} = T$, mg acts downward and, from the FBD, Newton's second law yields:

$$2T - mg - T \,=\, 0 \quad \Rightarrow \quad \boxed{T \,=\, mg.}$$

This time, T is provided by the force exerted from below, F, and thus the person pulling from below must exert a force $F = T = mg$, the full weight of m.

Alternate solution: One could also construct a control surface excluding the rope between points C and D, since this could be done without cutting off any body parts of m! Done correctly, this will give the same answer ($F = T = mg$) even if the FBD is different.

Problem 6.

Solution: a) Separating the variables, we get:

$$\frac{dy}{dx} = \frac{xy}{x^2+1} \quad \Rightarrow \quad \frac{dy}{y} = \frac{x\,dx}{x^2+1} = \frac{1}{2}\frac{d(x^2+1)}{x^2+1},$$

which we can integrate directly:

$$\int \frac{dy}{y} = \ln y = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\ln(x^2+1) + c,$$

where c is the combined constant of integration from both integrals. Solving for y, we get:

$$y = e^{\ln(x^2+1)^{1/2}+c} = e^c\sqrt{x^2+1} \quad \Rightarrow \quad \boxed{y(x) = C\sqrt{x^2+1},}$$

where $C = e^c$ is a "constant of integration".

b) Separating the variables, we get:

$$\frac{dy}{dx} = -xe^{-x}e^{-y} \quad \Rightarrow \quad e^y dy = -xe^{-x}dx,$$

which can be integrated directly:

$$\int e^y dy = e^y = -\int xe^{-x}dx = xe^{-x} - \int e^{-x}dx = xe^{-x} + e^{-x} + c$$

solving the second integral by parts, and where c is the combined constant of integration. Solving for y, we get:

$$\boxed{y(x) = \ln\big((x+1)e^{-x} + c\big).}$$

c) Following the hint, rewrite the ODE as,

$$x\frac{dy}{dx} + y = \frac{d(xy)}{dx} = \frac{dz}{dx} = 1 - x,$$

where $z = xy$. In terms of z, this ODE *is* separable, and we write:

$$dz = (1-x)dx \quad \Rightarrow \quad z = xy = x - \frac{x^2}{2} + c,$$

where c is the combined constant of integration. Solving for y, we get:

$$\boxed{y(x) = \frac{c}{x} + 1 - \frac{x}{2}.}$$

If you keep open to the possibility of a variable substitution such as in part c, you'll find many first order ODEs are separable, even if they don't look like they are at first blush.

Problem 7.

Solution: First, note that for forces that are functions only of x, and where at $x=0$, $v=0$, we have:

$$F(x) = ma = m\frac{dv}{dt} = m\frac{dx}{dt}\frac{dv}{dx} = mv\frac{dv}{dx} \quad\Rightarrow\quad \frac{1}{m}F(x)dx = v\,dv$$

$$\Rightarrow\quad \frac{1}{m}\int_0^x F(x')dx' = \int_{v(0)}^{v(x)} v'dv' = \frac{1}{2}\left(v^2(x) - \cancelto{0}{v^2(0)}\right)$$

$$\Rightarrow\quad v(x) = \sqrt{\frac{2}{m}\int_0^x F(x')dx'}. \tag{1}$$

This is a variation of equation (1.7.9) in the class notes with $E - U(x) = K$ replaced with $K = K_0 + \int_0^x F(x')dx'$ and $K_0 = 0$ (equation 1.7.5 in the class notes). Putting equation (1) in this form makes it more practical for solving this problem. To wit,

a) $F_x = F_0 + cx \quad\Rightarrow\quad v(x) = \sqrt{\frac{2}{m}\int_0^x (F_0 + cx')dx'} = \boxed{\sqrt{\frac{2}{m}\left(F_0 x + \tfrac{1}{2}cx^2\right)}}$

b) $F_x = F_0 e^{-cx} \quad\Rightarrow\quad v(x) = \sqrt{\frac{2}{m}\int_0^x F_0 e^{-cx'}dx'} = \boxed{\sqrt{\frac{2F_0}{cm}\left(1 - e^{-cx}\right)}}$

c) $F_x = F_0\cos(cx) \quad\Rightarrow\quad v(x) = \sqrt{\frac{2}{m}\int_0^x F_0\cos(cx')dx'} = \boxed{\sqrt{\frac{2F_0}{cm}\sin cx}}$

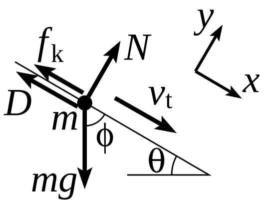
Problem 8.

Solution: a) From equation (2), the *differential form* of the work-kinetic theorem,

$$\sum dW = dK \quad\Rightarrow\quad \sum\frac{dW}{dt} = \sum P = \frac{dK}{dt} = 0, \tag{3}$$

since the rate of change of kinetic energy for an object moving at constant speed, $v_{\rm t}$, is zero.

Now, from the FBD, the four forces at play are mg, N, $f_{\rm k}$, and D, each in principle generating power. Thus, equation (3) becomes:



$$\sum P = P_{mg} + P_N + P_{f_{\rm k}} + P_D = 0$$

$$\Rightarrow\quad P_D = -P_{mg} - P_N - P_{f_{\rm k}}, \tag{4}$$

and it remains to calculate the power generated by each force on the RHS. To that end, and with x pointing in the direction of v_t as shown in the FBD, we use equation (1) to find:

$$\begin{aligned} P_{mg} &= (mg)_x v_\mathrm{t} = (mg\cos\phi)v_\mathrm{t} = mgv_\mathrm{t}\sin\theta && \text{(You need to know your trig!);} \\ P_N &= N_x v_\mathrm{t} = 0 && (\text{since } \vec{N} \perp \vec{v}_\mathrm{t}); \\ P_{f_\mathrm{k}} &= (f_\mathrm{k})_x v_\mathrm{t} = -\mu_\mathrm{k} N v_\mathrm{t} = -\mu_\mathrm{k} mg\cos\theta\, v_\mathrm{t} && (\text{since } \vec{f}_\mathrm{k} \propto -\hat{\imath} \text{ and } N = mg\cos\theta). \end{aligned}$$

Thus, equation (4) becomes:

$$P_D = -mgv_\mathrm{t}\sin\theta + \mu_\mathrm{k} mg\cos\theta\, v_\mathrm{t} \quad \Rightarrow \quad \boxed{P_D = mv_\mathrm{t} g(\mu_\mathrm{k}\cos\theta - \sin\theta).}$$

b) For numerical values, we have from above,

$$\begin{aligned} P_{mg} &= mv_\mathrm{t} g\sin\theta = (500)(9.81)(\sin 20^\circ) \sim \underline{\underline{1.68\times 10^3\,\mathrm{W} \sim 2.25\,\mathrm{hp};}} \\ P_{f_\mathrm{k}} &= -\mu_\mathrm{k} mv_\mathrm{t} g\cos\theta = -(0.15)(500)(9.81)(\cos 20^\circ) \sim \underline{\underline{-691.\,\mathrm{W} \sim -0.927\,\mathrm{hp};}} \\ P_D &= (500)(9.81)\big[(0.15)(\cos 20^\circ) - (\sin 20^\circ)\big] \sim \underline{\underline{-986.\,\mathrm{W} \sim -1.32\,\mathrm{hp},}} \end{aligned}$$

all accurate to three significant figures.

The sign on the power ($+$ for P_{mg}, $-$ for P_{f_k} and P_D) indicates whether the power generated by the force works to accelerate ($+$) or decelerate ($-$) the motion. That is, whether the component of $\vec{F}$ in the direction of $\vec{v}$ is parallel to ($+$) or antiparallel to ($-$) $\vec{v}$.

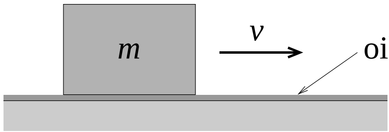
Problem 9.

Solution: a) $F = m\ddot{x} = m\dfrac{dv}{dt} = m\dfrac{dx}{dt}\dfrac{dv}{dx} = mv\dfrac{dv}{dx} = m\dfrac{\alpha}{x}\dfrac{d}{dx}\left(\dfrac{\alpha}{x}\right) = -m\dfrac{\alpha}{x}\dfrac{\alpha}{x^2}$

$$\Rightarrow \quad \boxed{F(x) = -\frac{m\alpha^2}{x^3}.}$$

b) $F = -cv^{3/2} = m\dot{v} = m\dfrac{dv}{dx}\dfrac{dx}{dt} = mv\dfrac{dv}{dx}$

$$\Rightarrow \quad \int_{v_0}^{0} \frac{v\,dv}{v^{3/2}} = -\frac{c}{m}\int_0^l dx,$$



where l is the total distance m slides, and thus to where $v = 0$. Solving the integrals, we get:

$$2v^{1/2}\Big|_{v_0}^{0} = -2v_0^{1/2} = -\frac{cl}{m} \quad \Rightarrow \quad \boxed{l = \frac{2m}{c}\sqrt{v_0}.}$$

Problem 10.

Solution: This is an application of problem 2.17 (F&C, ed. 7). We have:

$$F \,=\, kvx \,=\, ma \,=\, m\frac{dv}{dt} \,=\, mv\frac{dv}{dx} \quad\Rightarrow\quad kx \,=\, m\frac{dv}{dx},$$

which is a separable first order ODE. Thus,

$$dv \,=\, \frac{k}{m}x\,dx \quad\Rightarrow\quad \int dv \,=\, \frac{k}{m}\int x\,dx \quad\Rightarrow\quad v(x) \,=\, \frac{k}{2m}x^2 + c, \tag{1}$$

where c is a constant of integration evaluated using *boundary* conditions. In this case, at $x = 0$, $v = v_0$, and equation (1) $\Rightarrow v_0 = c$. Thus,

$$v(x) \,=\, \frac{dx}{dt} \,=\, \frac{k}{2m}x^2 + v_0.$$

Once again, we have a separable first order ODE, and we can write:

$$\frac{dx}{v_0 + kx^2/(2m)} \,=\, dt \quad\Rightarrow\quad \frac{2m}{k}\int\frac{dx}{x^2+a^2} \,=\, \int dt,$$

where $a^2 \equiv 2mv_0/k$. Letting $x = a\tan\phi$ and thus $dx = a\sec^2\!\phi\,d\phi$, we get,

$$\frac{2m}{k}\int\frac{\cancel{a}\,\cancel{\sec^2\!\phi}}{a^{\cancel{2}}\cancel{(\tan^2\phi+1)}}\,d\phi \,=\, \frac{2m}{ka}\int d\phi \,=\, \int dt \quad\Rightarrow\quad \sqrt{\frac{2m}{kv_0}}\,\phi \,=\, t + d,$$

where d is another constant of integration, this time evaluated from *initial* conditions. In particular, at $t = 0$, $x = 0$ and thus $d = 0$, and we have:

$$\sqrt{\frac{2m}{kv_0}}\tan^{-1}\!\left(\frac{x}{a}\right) \,=\, t \quad\Rightarrow\quad \tan^{-1}\!\left(x\sqrt{\frac{k}{2mv_0}}\right) \,=\, \sqrt{\frac{kv_0}{2m}}\,t$$

$$\Rightarrow\quad \boxed{x(t) \,=\, \sqrt{\frac{2mv_0}{k}}\tan\!\left(\sqrt{\frac{kv_0}{2m}}\,t\right).}$$

Problem 11.

Solution: a) Equation (1) includes a first and second derivative of y but not y itself. Thus, the most obvious solution to equation (1) is a constant, $y = c$, for then $y' = y'' = 0$, and equation (1) is satisfied trivially. So what constant do we choose? It doesn't matter; Occam might suggest $y_1(x) = 1$, so let's go with that.

As for the second solution, it may be a bit more obvious if we let $z = y'$ and rewrite equation (1) as:

$$z' + 2z \;=\; 0 \quad \Rightarrow \quad z' \;=\; -2z,$$

and ask the question: *What function, $z(x)$, has a first derivative equal to minus itself times 2?* Answer: $z(x) = e^{-2x}$, for then,

$$\frac{dz}{dx} \;=\; \frac{d}{dx}e^{-2x} \;=\; -2e^{-2x} \;=\; -2z.$$

But if $z(x) = e^{-2x}$, then,

$$y(x) \;=\; \int z(x)\,dx \;=\; -\frac{e^{-2x}}{2},$$

and $-\frac{1}{2}e^{-2x}$ also solves equation (1). But if $-\frac{1}{2}e^{-2x}$ solves equation (1), so will e^{-2x} (leading constants don't matter in a homogeneous ODE—try it!), and our two solutions are:

$$\boxed{y_1(x) \;=\; 1} \qquad \text{and} \qquad \boxed{y_2(x) \;=\; e^{-2x}.} \tag{2}$$

Evidently, there is no constant, α, such that $y_1 = \alpha y_2$, and the two solutions in equation (2) are linearly independent.

b) The general solution to equation (1) is a linear combination of the two linearly independent solutions in equations (2). That is,

$$\boxed{y(x) \;=\; Ay_1(x) + By_2(x) \;=\; A + Be^{-2x},} \tag{3}$$

where A and B are constants (independent of x).

c) Since one of the boundary conditions is applied to y', we first differentiate equation (3) to get:

$$y'(x) \;=\; -2Be^{-2x}. \tag{4}$$

Then, setting $y(0) = 1$ and $y'(0) = -1$, we get:

$$y(0) \;=\; A + B \;=\; 1 \quad \text{and} \quad y'(0) \;=\; -2B \;=\; -1 \quad \Rightarrow \quad A \;=\; B \;=\; \frac{1}{2},$$

which, when substituted into equation (3), gives:

$$y(x) \;=\; \frac{1}{2}\big(1 + e^{-2x}\big) \;=\; e^{-x}\underbrace{\frac{e^{x} + e^{-x}}{2}}_{\cosh x} \quad \Rightarrow \quad \boxed{y(x) \;=\; e^{-x}\cosh x,}$$

as desired.

Incidentally, equation (1) could also be solved by breaking it up into two first order, separable ODEs. Letting, as we did above, $z = y'$, equation (1) becomes the 1st order ODE:

$$\frac{dz}{dx} + 2z = 0 \quad \Rightarrow \quad \frac{dz}{z} = -2dx \quad \Rightarrow \quad \int \frac{dz}{z} = -2\int dx \quad \Rightarrow \quad \ln z = -2x + \beta,$$

where β is a constant of integration. Thus,

$$z(x) = e^{-2x+\beta} = be^{-2x},$$

where $b = e^{\beta}$. But $y' = z$, and we have the second separable 1st order ODE:

$$\frac{dy}{dx} = be^{-2x} \quad \Rightarrow \quad dy = be^{-2x}dx \quad \Rightarrow \quad \int dy = b\int e^{-2x}dx \quad \Rightarrow \quad \boxed{y(x) = A + Be^{-2x},} \quad (5)$$

where $B = -b/2$ and A is a second constant of integration.

Equation (5) is identical to equation (3), and thus if you followed this path, you would have answered part b before part a. To answer part a, you would have had to glean the two independent solutions from equation (5), namely $y_1(x) = 1$ and $y_2(x) = e^{-2x}$. Part c then follows directly from equation (5) as above.

Problem 12.

Solution: a) The centre of a guitar string moves back and forth like a simple harmonic oscillator with amplitude $x_0 = 2.00 \times 10^{-3}\,\mathrm{m}$ and angular frequency $\omega_0 = 2\pi f = 1{,}024\,\pi\,\mathrm{rad\,s^{-1}}$.

Now, from equation (2.2.5) in the class notes, we have:

$$x(t) = x_0 \cos\omega_0 t \quad \Rightarrow \quad \dot{x}(t) = -x_0\omega_0 \sin\omega_0 t \quad \Rightarrow \quad \ddot{x}(t) = -x_0\omega_0^2 \cos\omega_0 t.$$

Since sin and cos range between ±1, the maximum velocity is given when $\sin\omega_0 t = \pm 1$ (and thus $\omega_0 t = \pm\pi/2$) and the maximum acceleration is given when $\cos\omega_0 t = \pm 1$ (and thus $\omega_0 t = 0$ or π). Thus,

$$v_{\max} = x_0\omega_0 = (2.00 \times 10^{-3})(1.024 \times 10^3)\pi \sim \underline{\underline{6.43\,\mathrm{m\,s^{-1}}}},$$

$$a_{\max} = x_0\omega_0^2 = (2.00 \times 10^{-3})(1.024 \times 10^3)^2\pi^2 \sim \underline{\underline{2.07 \times 10^4\,\mathrm{m\,s^{-2}}}}.$$

b) Starting this time from equation (2.2.3) in the class notes,

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t \quad \Rightarrow \quad \dot{x}(t) = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t,$$

we apply boundary conditions to find A and B. In this case, at $t = 0$,

$$x(0) = 0.250 = A\cancelto{1}{\cos(0)} + B\cancelto{0}{\sin(0)} = A;$$

$$\dot{x}(0) \;=\; 0.100 \;=\; -A\omega_0 \cancelto{0}{\sin(0)} + B\omega_0 \cancelto{1}{\cos(0)} \;=\; B\omega_0 \quad\Rightarrow\quad B \;=\; \frac{0.100}{2\pi f} \;=\; \frac{5.00\times 10^{-3}}{\pi}\,\text{m}.$$

Thus,

$$\boxed{x(t) \;=\; 0.250\cos(20\pi t) + \frac{5.00\times 10^{-3}}{\pi}\sin(20\pi t),}$$

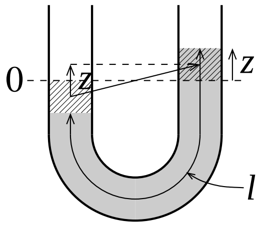
or,

$$x(t) \;=\; 0.250\cos(62.8t) + 1.59\times 10^{-3}\sin(62.8t).$$

Problem 13.

Solution: a) Let $z = 0$ be the equilibrium position when the levels in both sides of the U-tube are equal, and consider the system when the water level in the right hand side is a distance $0 < z \leq z_0$ above equilibrium (inset).

Defining $U(0) = 0$, the potential energy of the system is as though water from the hatched region on the left were raised a distance z. Thus, if A is the uniform cross sectional area of the U-tube,



$$U(z) \;=\; mgh \;=\; (\rho A z)gz \;=\; \rho A g z^2,$$

where ρ is the density of water, and Az is the volume of water raised a distance z. The fact that $U(z)$ is quadratic in displacement, z, is enough to identify this system as a simple harmonic oscillator.

To find the period of oscillation, we must find the kinetic energy. Here, K is the energy of motion of the entire length of the water column. Further, because the cross sectional area of the U-tube is constant, every point in the water column moves with the same velocity, $v = \dot{z}$, the speed of the rising and falling water levels on the left and right side. Thus,

$$K(\dot{z}) \;=\; \tfrac{1}{2}mv^2 \;=\; \tfrac{1}{2}(\rho A l)\dot{z}^2,$$

where this time, m is the mass of the entire column of water of length l.

Conserving energy, we have:

$$E \;=\; U(z) + K(\dot{z}) \;=\; \rho A g z^2 + \frac{\rho A l}{2}\dot{z}^2,$$

which has the form of equation (2.3.1) in the lecture notes. Thus, ω_0^2 is the ratio of the coefficients of the z^2 and $\dot{z}^2$ terms, and we get:

$$\omega_0^2 \;=\; \frac{\cancel{\rho A}g}{\frac{1}{2}\cancel{\rho A}l} \;=\; \frac{2g}{l} \quad\Rightarrow\quad \boxed{T \;=\; \frac{2\pi}{\omega_0} \;=\; 2\pi\sqrt{\frac{l}{2g}} \;=\; \pi\sqrt{\frac{2l}{g}},}$$

as desired.

b) Since $\omega_0^2 = \dfrac{2g}{l} = \dfrac{k}{m}$ for an effective spring constant, k, then,

$$k = \frac{2gm}{l} = \frac{2g\rho Al}{l} \quad\Rightarrow\quad \boxed{k = 2g\rho A.}$$

Problem 14.

Solution: a) From the discussion on vertical oscillators in §2.2 of the class notes, if we measure displacements from the equilibrium point of m hanging on the wire (length $l + \delta l \approx l$), then we need only consider the unbalanced force caused by the additional distortion $-z_0 \leq z \leq z_0$, as shown in the inset. Thus, from the FBD,

$$F = m\ddot{z} = SA = -Y\Sigma A = -Y\frac{z}{l+\delta l}A \approx -\frac{YA}{l}z,$$

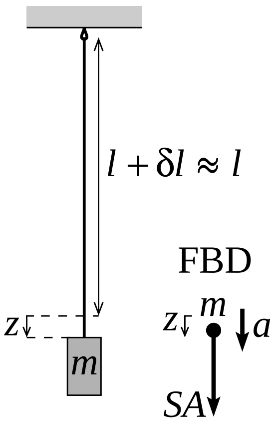
using equation (2) to set $F = SA$, equation (3) to set $S = -Y\Sigma$, and equation (1) for Σ. Thus,

$$\boxed{\ddot{z} = -\frac{YA}{ml}z,}$$

the equation of motion for a simple harmonic oscillator. The frequency and period of oscillation are therefore given by:

$$\omega_0^2 = \frac{YA}{ml} \quad\Rightarrow\quad \boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{ml}{YA}},}$$

which is equation (4).



b) A 12-gauge wire has a diameter $d = 2.05\,\mathrm{mm}$, and thus a cross sectional area,

$$A = \pi\frac{d^2}{4} = \pi\frac{(2.05)^2}{4} \sim 3.301\,\mathrm{mm}^2 = 3.301\times 10^{-6}\,\mathrm{m}^2,$$

retaining four significant figures for an intermediate result. Thus,

$$T = 2\pi\sqrt{\frac{(1.00)(1.00)}{(1.17\times 10^{11})(3.301\times 10^{-6})}} \sim 0.0101\,\mathrm{s},$$

or about 99 Hz (oscillations per second). It is for this reason that demonstrating such oscillations is difficult in a classroom setting.

c) When m is hanging from the spring in equilibrium, its weight is balanced by the restoring force of the stretched wire. Thus,

$$mg + SA = 0 \quad\Rightarrow\quad mg = -SA = Y\Sigma A = Y\frac{\delta l}{l}A \quad\Rightarrow\quad \frac{ml}{YA} = \frac{\delta l}{g}.$$

Substituting this into equation (4), we get:

$$\boxed{T = 2\pi\sqrt{\frac{\delta l}{g}},}$$

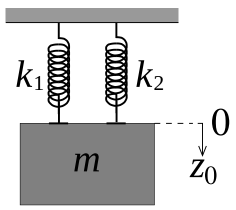
which is the period of a simple pendulum of length δl (equation 2.2.7 in the class notes).

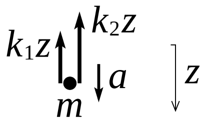
Problem 15.

Solution: a) Let the equilibrium position of m be $z = 0$. Then, as shown in §2.2, the forces to establish equilibrium (mg, $k_1 z_e$, $k_2 z_e$) balance and can be omitted from the free body diagram.

If m is pulled down a distance z_0 and released, FBD 1 below shows the relevant forces when m is at $-z_0 \leq z \leq z_0$. Thus, from Newton's 2nd law, we have:

$$-k_1 z - k_2 z = m\ddot{z} \quad \Rightarrow \quad \ddot{z} = -\frac{k_1 + k_2}{m} z,$$





FBD 1

a second order ODE describing a simple harmonic oscillator with frequency:

$$\boxed{\omega_0 = \sqrt{\frac{k_1 + k_2}{m}}.}$$

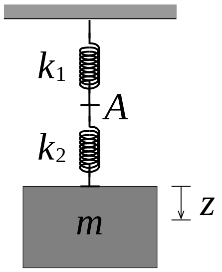
If we were to replace the two springs with a single spring of spring constant k_{par} without changing the angular frequency, then:

$$\omega_0 = \sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{k_{\text{par}}}{m}} \quad \Rightarrow \quad k_{\text{par}} = k_1 + k_2.$$

Thus, springs in parallel combine by *direct addition* (like electrical resistors in series).

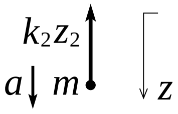
b) Next, if the two springs are attached to m in series, the displacement of m, namely z, is distributed over the two springs. Let k_1 and k_2 be stretched (compressed) by z_1 and z_2 respectively. Then,

$$z_1 + z_2 = z \quad \Rightarrow \quad z_1 = z - z_2. \tag{1}$$

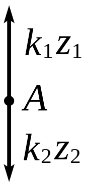


Further, only one spring (k_2) is in direct contact with m. Thus, the FBD for m (FBD 2) includes $k_2 z_2$ only (equilibrium forces ignored), and we have:

$$-k_2 z_2 = m\ddot{z}. \tag{2}$$



FBD 2



FBD 3

To incorporate the effect of spring k_1, we consider point A where the two springs join. Here, if both springs are stretched, k_1 pulls upward on A while k_2 pulls downward, as shown on FBD 3, and Newton's 2nd law gives:

$$k_2 z_2 - k_1 z_1 = 0 \quad \text{(since } A \text{ has no mass)} \tag{3}$$

Now for the algebra. Substitute equation (1) into equation (3) to get:

$$k_2 z_2 - k_1 z + k_1 z_2 = 0 \quad \Rightarrow \quad z_2 = \frac{k_1}{k_1 + k_2} z. \tag{4}$$

Then, substituting equation (4) into equation (2) gives us:

$$-\frac{k_1 k_2}{k_1 + k_2} z = m\ddot{z} \quad \Rightarrow \quad \boxed{\ddot{z} = -\frac{1}{m}\frac{k_1 k_2}{k_1 + k_2} z,}$$

the equation for a simple harmonic oscillator, with frequency,

$$\boxed{\omega_0 = \sqrt{\frac{1}{m}\frac{k_1 k_2}{k_1 + k_2}}.}$$

If we were to replace the two springs with a single spring of spring constant k_{ser} without changing the angular frequency, then:

$$\omega_0 = \sqrt{\frac{1}{m}\frac{k_1 k_2}{k_1 + k_2}} = \sqrt{\frac{k_{\text{ser}}}{m}} \quad \Rightarrow \quad k_{\text{ser}} = \frac{k_1 k_2}{k_1 + k_2} \quad \Rightarrow \quad \frac{1}{k_{\text{ser}}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

Thus, springs in series *add in reciprocal* (like electrical resistors in parallel).

Problem 16.

Solution: For an underdamped oscillator with initial amplitude z_0, the amplitude as a function of time is given by (equation 2.4.11 from the class notes),

$$A(t) = z_0 e^{-\gamma t}.$$

Thus, time for amplitude to fall by 1/2, $t_{1/2}$, is given by:

$$A(t_{1/2}) = \frac{z_0}{2} = z_0 e^{-\gamma t_{1/2}} \quad \Rightarrow \quad \gamma t_{1/2} = \ln 2 \quad \Rightarrow \quad t_{1/2} = \frac{\ln 2}{\gamma}.$$

Now, the number of complete periods for the amplitude to fall to $z_0/2$ is 10, and thus,

$$\frac{t_{1/2}}{T_0} = 10 = \frac{\ln 2}{\gamma}\frac{1}{\pi}\sqrt{\frac{g}{2l}} \quad \Rightarrow \quad \boxed{\gamma = \frac{\ln 2}{10\pi}\sqrt{\frac{g}{2l}},}$$

using equation (1) for T_0.

Problem 17.

Solution: As was done in class, we differentiate equation (1) to get:

$$\dot{x}(t) \;=\; -\gamma e^{-\gamma t}\left(Ae^{qt} + Be^{-qt}\right) + e^{-\gamma t} q \left(Ae^{qt} - Be^{-qt}\right), \tag{2}$$

and apply the new initial conditions to equations (1) and (2) to find A and B. To that end,

$$x(0) \;=\; A + B \;=\; 0 \qquad \text{and} \qquad \dot{x}(0) \;=\; -\gamma\cancelto{0}{(A+B)} + q(A-B) \;=\; v_0$$

$$\Rightarrow \quad A \;=\; -B \;=\; \frac{v_0}{2q}$$

Thus, equation (1) becomes:

$$x(t) \;=\; e^{-\gamma t}\frac{v_0}{2q}(e^{qt} - e^{-qt}) \;=\; \frac{v_0}{q}\, e^{-\gamma t}\sinh qt \quad (q \neq 0), \tag{3}$$

using the definition of the hyperbolic sine given by equation (2.4.8) in the class notes.

As done in class, we consider four cases.

Case 1: For $q > 0 \in \mathbb{R}$, $\gamma > \omega_0$ and the system is *over-damped*. This is described by equation (3) as written.

Case 2: For $q = 0$, $\gamma = \omega_0$ and the system is *critically damped*. For this, we must examine equation (3) in the limit as $q \to 0$. Thus,

$$\lim_{q\to 0} x(t) \;=\; v_0 e^{-\omega_0 t}\lim_{q\to 0}\frac{\sinh qt}{q} \;=\; v_0 e^{-\omega_0 t}\underbrace{\lim_{q\to 0}\frac{t\cosh qt}{1}}_{\text{l'Hôpital}} \;=\; v_0 t e^{-\omega_0 t}, \tag{4}$$

since $\cosh 0 = 1$.

Case 3: For $\gamma < \omega_0$, $q \in \mathbb{I}$ and the system is *underdamped*. For this, we let,

$$\omega_{\mathrm{d}} \;=\; \sqrt{\omega_0^2 - \gamma^2} \;=\; q/i,$$

where $\omega_{\mathrm{d}} \in \mathbb{R}$ is the natural oscillation frequency of the underdamped oscillator. Thus, $q = i\omega_{\mathrm{d}}$ and equation (3) becomes:

$$x(t) \;=\; \frac{v_0}{i\omega_{\mathrm{d}}}\, e^{-\gamma t}\sinh i\omega_{\mathrm{d}} t \;=\; \frac{v_0}{\omega_{\mathrm{d}}}\, e^{-\gamma t}\sin\omega_{\mathrm{d}} t, \tag{5}$$

using one of equations (2.4.9) in the class notes.

Case 4: For $\gamma = 0$, $\omega_{\mathrm{d}} = \omega_0$ and the system is *undamped*, in which case,

$$\lim_{\gamma\to 0} x(t) \;=\; \frac{v_0}{\omega_0}\sin\omega_0 t. \qquad (6).$$

Bringing all four cases together, we have:

$$\boxed{x(t) = v_0 \begin{cases} e^{-\gamma t}\dfrac{\sinh qt}{q}, & \gamma > \omega_0 \text{ (overdamped);} \\ te^{-\omega_0 t}, & \gamma = \omega_0 \text{ (critically damped);} \\ e^{-\gamma t}\dfrac{\sin \omega_\mathrm{d} t}{\omega_\mathrm{d}}, & \gamma < \omega_0 \text{ (underdamped);} \\ \dfrac{\sin \omega_0 t}{\omega_0}, & \gamma = 0 \text{ (undamped),} \end{cases}}$$

as desired.

Problem 18.

Solution: a) From problem 3.9, we have that the ratio of successive minima of a damped oscillator is $e^{-\gamma T_\mathrm{d}}$, where $T_\mathrm{d} = 2\pi/\omega_\mathrm{d}$ is the period of oscillation. Thus,

$$e^{-\gamma T_\mathrm{d}} = \frac{1}{2} \quad \Rightarrow \quad \gamma T_\mathrm{d} = \frac{\gamma}{f_\mathrm{d}} = \ln 2 \quad \Rightarrow \quad \gamma = f_\mathrm{d} \ln 2, \tag{1}$$

where f_d is the frequency of the damped oscillator in Hz. Now, from the class notes, we have:

$$\omega_\mathrm{d} = \sqrt{\omega_0^2 - \gamma^2} \quad \Rightarrow \quad \omega_0 = \sqrt{\omega_\mathrm{d}^2 + \gamma^2} = \sqrt{\omega_\mathrm{d}^2 + (f_\mathrm{d}\ln 2)^2} = \omega_\mathrm{d}\sqrt{1 + \left(\frac{\ln 2}{2\pi}\right)^2},$$

since $f_\mathrm{d} = \omega_\mathrm{d}/2\pi$. Thus,

$$f_0 = \frac{\omega_0}{2\pi} = \frac{\omega_\mathrm{d}}{2\pi}\sqrt{1 + \left(\frac{\ln 2}{2\pi}\right)^2} \sim f_\mathrm{d}(1.00607) \sim \underline{\underline{100.6\,\mathrm{Hz}}},$$

for $f_\mathrm{d} = 100\,\mathrm{Hz}$. Note that a damping coefficient sufficient to decrease the amplitude of oscillation by a factor of *two* after each cycle only changes the oscillation frequency by 0.6%. Thus, for the most part, $\omega_\mathrm{d} \approx \omega_0$ is a pretty good approximation.

b) From equation (2.4.16) in the class notes,

$$Q_\mathrm{d} = \frac{\omega_d}{2\gamma} = \frac{2\pi f_\mathrm{d}}{2 f_\mathrm{d}\ln 2} = \frac{\pi}{\ln 2} \sim \underline{\underline{4.53}},$$

using equation (1) for γ.

Problem 19.

Solution: a) Let $t_{\rm cd}$ be the time it takes for the critically damped oscillator to get to within 1% of its equilibrium position. That is, from equation (2.4.4) in the class notes,

$$x(t_{\rm cd}) \;=\; 0.01x_0 \;=\; x_0 e^{-\gamma t_{\rm cd}}(1+\gamma t_{\rm cd})$$

$$\Rightarrow \quad e^{-\gamma t_{\rm cd}}(1+\gamma t_{\rm cd}) \;=\; 0.01,$$

a transcendental equation we must solve with a root finder. My Hewlett-Packard 15C programmable calculator (bought in 1982!) is perfectly suited for this task, and I find,

$$\gamma t_{\rm cd} \;\sim\; 6.638 \text{ (unitless)}.$$

But, for critical damping, $\gamma = \omega_0 = 2\pi f_0 \;=\; 10\pi$, and thus,

$$10\pi t_{\rm cd} \;\sim\; 6.638 \quad\Rightarrow\quad t_{\rm cd} \;\sim\; \underline{\underline{0.211\,\text{s}}}.$$

b) Let $t_{\rm od}$ be the time it takes for the over-damped oscillator to get to within 1% of its equilibrium position. That is, from equation (2.4.10) in the class notes,

$$x(t_{\rm od}) \;=\; 0.01x_0 \;=\; x_0 e^{-\gamma t_{\rm od}}\left(\cosh q t_{\rm od} + \frac{\gamma}{q}\sinh q t_{\rm od}\right), \tag{1}$$

where $q = \sqrt{\gamma^2 - \omega_0^2} \;=\; \sqrt{3}\omega_0$, for $\gamma = 2\omega_0$. Thus, equation (1) becomes:

$$e^{-2\omega_0 t_{\rm od}}\left(\cosh\sqrt{3}\omega_0 t_{\rm od} + \frac{2}{\sqrt{3}}\sinh\sqrt{3}\omega_0 t_{\rm od}\right) \;=\; 0.01,$$

another transcendental equation for which my HP15C is equally well-suited! For this, I find,

$$\omega_0 t_{\rm od} \;\sim\; 17.465 \quad\Rightarrow\quad t_{\rm od} \;\sim\; \frac{17.465}{10\pi} \;\sim\; \underline{\underline{0.556\,\text{s}}}.$$

c) For an underdamped galvanometer, we have from equation (2.4.10) in the class notes,

$$x(t) \;=\; \underbrace{x_0 e^{-\gamma t}}_{A(t)}\left(\cos\omega_{\rm d}t + \frac{\gamma}{\omega_{\rm d}}\sin\omega_{\rm d}t\right),$$

where the amplitude, $A(t)$, is given by the underbrace, with the remaining time-dependence oscillatory. Thus, if $t_{\rm ud}$ is the time for the amplitude to die down to 1% of its original value,

$$A(t_{\rm ud}) \;=\; 0.01x_0 \;=\; x_0 e^{-\gamma t_{\rm ud}} \quad\Rightarrow\quad e^{-\gamma t_{\rm ud}} \;=\; 0.01,$$

where we can now solve for $t_{\rm ud}$ directly. Thus, with $\gamma = \omega_0/5$,

$$-\frac{\omega_0}{5}t_{\rm ud} \;=\; \ln(0.01) \quad\Rightarrow\quad t_{\rm ud} \;=\; \frac{5}{\omega_0}\ln(100) \;\sim\; \underline{\underline{0.733\,\text{s}}}.$$

Now,

$$\omega_{\rm d} \;=\; \sqrt{\omega_0^2 - \gamma^2} \;=\; \omega_0 \frac{\sqrt{24}}{5}$$

$$\Rightarrow \quad \omega_{\rm d} t_{\rm ud} \;=\; \omega_0 \, \frac{\sqrt{24}}{5} \, \frac{5 \ln(100)}{\omega_0} \;=\; \sqrt{24} \ln(100) \;\sim\; 22.56 \;\sim\; 7.18\pi,$$

or about $\underline{\underline{\text{3.59 complete oscillations}}}$.

Note that in both parts b and c, the time for the needle to get to within 1% of its "reading" (equilibrium) is longer than in part a, where the galvanometer is critically damped.

Oh yeah, and the speedometer? It's from my very first car:
a 1978 Firebird!

Congrats to Jared Park (and his dad) who identified it!

Problem 20.

Solution: For each part, a and b, we first solve the homogeneous version of equation (1),

$$y_{\rm h}''(x) + 5y_{\rm h}'(x) + 4y_{\rm h}(x) \;=\; 0, \tag{2}$$

by "trial exponentials". Thus, substitute the trial solution e^{rx} into equation (2) to get:

$$r^2 \cancel{e^{rx}} + 5r\cancel{e^{rx}} + 4\cancel{e^{rx}} \;=\; 0 \quad \Rightarrow \quad r^2 + 5r + 4 \;=\; (r+1)(r+4) \;=\; 0,$$

factoring the quadratic (or we could have used the quadratic formula). Thus, $r = -1$ or -4 and our two linearly independent solutions to equation (2) are:

$$y_1(x) \;=\; e^{-x} \qquad \text{and} \qquad y_2(x) \;=\; e^{-4x},$$

from which we construct the general solution to (the homogeneous) equation (2):

$$y_{\rm h}(x) \;=\; Ae^{-x} + Be^{-4x}, \tag{3}$$

where A and B are constants to be determined from boundary conditions.

a) For $f(x) = 2$, equation (1) becomes:

$$y''(x) + 5y'(x) + 4y(x) \;=\; 2, \tag{4}$$

for which we search for *any* solution by inspection. Since all coefficients are constant, try $y_{\rm p} = $constant $\Rightarrow y_{\rm p}'' = y_{\rm p}' = 0$, and equation (4) becomes:

$$4y_{\rm p} \;=\; 2 \quad \Rightarrow \quad y_{\rm p} \;=\; \frac{1}{2}.$$

The general solution to equation (1) is thus,

$$y(x) \;=\; y_{\mathrm{h}}(x) + y_{\mathrm{p}} \;=\; Ae^{-x} + Be^{-4x} + \frac{1}{2} \quad\Rightarrow\quad y'(x) \;=\; -Ae^{-x} - 4Be^{-4x}.$$

Applying the given boundary conditions, we get:

$$y(0) \;=\; A + B + \frac{1}{2} \;=\; 1 \qquad \text{and} \qquad y'(0) \;=\; -A - 4B \;=\; 0.$$

Solving for A and B, we get:

$$A \;=\; \frac{2}{3}, \quad B \;=\; -\frac{1}{6} \quad\Rightarrow\quad \boxed{y(x) \;=\; \frac{2}{3}e^{-x} - \frac{1}{6}e^{-4x} + \frac{1}{2},} \tag{5}$$

is the specific solution to equation (1) for the given boundary conditions and $f(x) = 2$.

b) For $f(x) = 5e^x$, equation (1) becomes:

$$y''(x) + 5y'(x) + 4y(x) \;=\; 5e^x, \tag{6}$$

for which we search for *any* solution by trial exponentials. Noting that if $y_{\mathrm{p}} \propto e^x$, all x-dependence cancels out, we are motivated to try $y_{\mathrm{p}}(x) = Ce^x$, in which case equation (6) becomes:

$$C\cancel{e^x} + 5C\cancel{e^x} + 4C\cancel{e^x} \;=\; 5\cancel{e^x} \quad\Rightarrow\quad 10C \;=\; 5 \quad\Rightarrow\quad C \;=\; \frac{1}{2}.$$

The general solution to equation (1) is thus,

$$y(x) \;=\; y_{\mathrm{h}}(x) + y_{\mathrm{p}}(x) \;=\; Ae^{-x} + Be^{-4x} + \frac{1}{2}e^x \quad\Rightarrow\quad y'(x) \;=\; -Ae^{-x} - 4Be^{-4x} + \frac{1}{2}e^x.$$

Applying the given boundary conditions, we get:

$$y(0) \;=\; A + B + \frac{1}{2} \;=\; 1 \qquad \text{and} \qquad y'(0) \;=\; -A - 4B + \frac{1}{2} \;=\; 0.$$

Solving for A and B, we get:

$$A \;=\; \frac{1}{2}, \quad B \;=\; 0 \quad\Rightarrow\quad \boxed{y(x) \;=\; \frac{1}{2}e^{-x} + \frac{1}{2}e^x \;=\; \cosh x,} \tag{7}$$

is the specific solution to equation (1) for the given boundary conditions and $f(x) = 5e^x$.

As an exercise, you should check to make sure that equations (5) and (7) do indeed solve equation (1) for the respective $f(x)$, and that in each case, $y(0) = 1$ and $y'(0) = 0$.

Problem 21.

Solution: a) From equation (2.6.12) in the class notes,

$$\omega_{\rm r} \;=\; \sqrt{\omega_0^2 - 2\gamma^2},$$

where $\omega_0^2 \;=\; \dfrac{k}{m}$ and $\gamma = \dfrac{b}{2m}$. Thus, from the numbers given,

$$\omega_0 \;=\; \sqrt{\frac{250.}{10.0}} \;=\; 5.00 \qquad \text{and} \qquad \gamma \;=\; \frac{60.0}{2(10.0)} \;=\; 3.00$$

$$\Rightarrow \quad \omega_{\rm r} \;=\; \sqrt{25.0 - 2(9.00)} \;=\; \underline{\underline{\sqrt{7}\ \sim\ 2.65\,{\rm rad\,s^{-1}}}}.$$

b) Start by noting that,

$$\omega_{\rm d} \;=\; \sqrt{\omega_0^2 - \gamma^2} \;=\; \sqrt{25.0 - 9.00} \;=\; 4.00.$$

Thus, from equations (2.6.13) and (2.6.15) in the class notes, we have,

$$A_{\rm max} \;=\; \frac{F_0}{b\,\omega_{\rm d}} \;=\; \frac{(48.0)}{(60.0)(4.00)} \;=\; \underline{\underline{0.200\,{\rm m}}},$$

$$\tan\phi_{\rm r} \;=\; \frac{\omega_{\rm r}}{\gamma} \;=\; \frac{\sqrt{7}}{3} \quad\Rightarrow\quad \phi_{\rm r} \;\sim\; \underline{\underline{0.723\,{\rm rad}}}.$$

Problem 22.

Solution: Here, we follow exactly what we did in class, with $F \propto \sin\omega t$ instead of $\cos\omega t$. Thus, we guess for the particular solution $x_{\rm p} \propto \sin\omega t$ and then, allowing for the force and displacement to be "out of phase" by a phase angle ϕ, choose,

$$x_{\rm p}(t) \;=\; A\sin(\omega t - \phi). \tag{1}$$

If this solves equation (2.6.1) from the class notes with the new driving force, namely,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x \;=\; \frac{F_0}{m}\sin\omega t,$$

then,

$$-A\omega^2\sin(\omega t - \phi) + 2\gamma A\omega\cos(\omega t - \phi) + \omega_0^2 A\sin(\omega t - \phi) \;=\; \frac{F_0}{m}\sin\omega t, \tag{2}$$

from which we solve for A and ϕ as functions of ω.

To this end, expanding out the trig functions in equation (2), we get:

$$(\omega_0^2 - \omega^2)A(\sin\omega t\cos\phi - \cos\omega t\sin\phi) + 2\gamma\omega A(\cos\omega t\cos\phi + \sin\omega t\sin\phi) \;=\; \frac{F_0}{m}\sin\omega t$$

$$\Rightarrow \quad \left[(\omega_0^2-\omega^2)A\cos\phi+2\gamma\omega A\sin\phi-\frac{F_0}{m}\right]\sin\omega t \;=\; \left[(\omega_0^2-\omega^2)A\sin\phi-2\gamma\omega A\cos\phi\right]\cos\omega t. \tag{3}$$

Since $\sin\omega t$ and $\cos\omega t$ are independent functions, the quantities in [ ] must be zero. Thus, on the RHS of equation (3):

$$(\omega_0^2-\omega^2)\cancel{A}\sin\phi-2\gamma\omega\cancel{A}\cos\phi \;=\; 0$$

$$\Rightarrow \quad \boxed{\tan\phi \;=\; \frac{2\gamma\omega}{\omega_0^2-\omega^2},} \tag{4}$$

which is equation (2.6.4) in the class notes. From this we can immediately write:

$$\sin\phi \;=\; \frac{2\gamma\omega}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}} \qquad \text{and} \qquad \cos\phi \;=\; \frac{\omega_0^2-\omega^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}, \tag{5}$$

which are equations (2.6.5) in the class notes.

On the LHS of equation (3), we have:

$$(\omega_0^2-\omega^2)A\cos\phi+2\gamma\omega A\sin\phi-\frac{F_0}{m} \;=\; 0$$

$$\Rightarrow \quad A \;=\; \frac{F_0/m}{(\omega_0^2-\omega^2)\cos\phi+2\gamma\omega\sin\phi} \;=\; \frac{\left[\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2\right]^{1/2}F_0/m}{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2},$$

using equations (5). Thus,

$$\boxed{A(\omega) \;=\; \frac{F_0/m}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}},} \tag{6}$$

which is equation (2.6.7) in the class notes.

Thus, $x_\mathrm{p}(t)$ in equation (1) is the steady-state solution for the damped harmonic oscillator driven by a force $F = F_0\sin\omega t$, with the phase between the force and displacement, ϕ, given by equation (4) and the amplitude of oscillation given by equation (6).

The point here is that sine and cosine are really the same function, just out of phase by $\pi/2$ radians. Thus, it makes no difference whether the driving force is proportional to $\sin\omega t$ or $\cos\omega t$; the amplitude and phase of the steady-state oscillation must be the same.

Problem 23.

Solution: a) As usual for a spring, the restoring force is $-kz$ (negative proportional to the distortion of the spring) and the damping force is $-b\dot{z}$ (negative proportional to the rate at which the spring is distorting). Thus, from Newton's second law, we have,

$$\sum F = -kz - b\dot{z} = ma, \tag{1}$$

where a is the acceleration as measured from an inertial frame of reference. Using O as the inertial frame, if y is the total distance between O and the bottom of the spring, then

$$y = \zeta + h + z \quad \Rightarrow \quad a = \ddot{y} = \frac{d^2}{dt^2}(\zeta + h + z) = \ddot{\zeta} + \ddot{z},$$

since $h =$ constant. Thus, with $b = 2m\gamma$ and $k = m\omega_0^2$, equation (1) becomes:

$$-\cancel{m}\omega_0^2 z - 2\gamma \cancel{m}\dot{z} = \cancel{m}(\ddot{\zeta} + \ddot{z}) \quad \Rightarrow \quad \boxed{\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = -\ddot{\zeta},} \tag{2}$$

as desired.

b) For $\zeta(t) = \zeta_0 \cos\omega t$, $\ddot{\zeta} = -\omega^2\zeta_0\cos\omega t$, and equation (2) becomes:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = \omega^2\zeta_0\cos\omega t,$$

which has exactly the same form as equation (2.6.1) from the class notes for a driven, damped oscillator with F_0/m replaced with $\omega^2\zeta_0$. Therefore, from equations (2.6.4) and (2.6.7) in the class notes, we can immediately write down,

$$\boxed{\tan\phi = \frac{2\omega\gamma}{\omega_0^2 - \omega^2}} \quad \text{and} \quad \boxed{A = \frac{\omega^2\zeta_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\gamma^2}}.} \tag{3}$$

c) Using the expression given,

$$Q_\text{d} = \frac{\omega_\text{d}}{2\gamma} = \frac{\sqrt{\omega_0^2 - \gamma^2}}{2\gamma} \quad \Rightarrow \quad 4\gamma^2 Q_\text{d}^2 = \omega_0^2 - \gamma^2 \quad \Rightarrow \quad \gamma = \frac{\omega_0}{\sqrt{4Q_\text{d}^2 + 1}}.$$

Thus, for $Q_\text{d} = 2$, we have:

$$\boxed{\gamma \approx \frac{\omega_0}{\sqrt{17}} \sim 0.243\omega_0.} \tag{4}$$

d) *i*) For $\omega = \frac{1}{2}\omega_0$ and using equation (4) for γ,

$$\frac{A}{\zeta_0} = \frac{\frac{1}{4}\cancel{\omega_0^2}}{\sqrt{\left(\frac{3}{4}\cancel{\omega_0^2}\right)^2 + \cancel{\omega_0^2}\left(\cancel{\omega_0^2}/17\right)}} = \frac{1/4}{\sqrt{9/16 + 1/17)}} \sim \underline{\underline{0.317}}.$$

$$\tan\phi \;=\; \frac{\cancel{\omega_0}(\cancel{\omega_0}/\sqrt{17})}{\frac{3}{4}\cancel{\omega_0^2}} \;=\; \frac{4}{3\sqrt{17}} \qquad\Rightarrow\qquad \phi \;=\; \tan^{-1}\left(\frac{4}{3\sqrt{17}}\right) \;\sim\; \underline{\underline{0.313\,\mathrm{rad}}},$$

and the displacement of m is within ~ 18° of being in phase with the ground.

ii) For $\omega = 2\omega_0$ and using equation (4) for γ,

$$\frac{A}{\zeta_0} \;=\; \frac{4\cancel{\omega_0^2}}{\sqrt{\left(-3\cancel{\omega_0^2}\right)^2 + 16\cancel{\omega_0^2}\left(\cancel{\omega_0^2}/17\right)}} \;=\; \frac{4}{\sqrt{9+16/17}} \;\sim\; \underline{\underline{1.27}}.$$

$$\tan\phi \;=\; \frac{4\cancel{\omega_0}(\cancel{\omega_0}/\sqrt{17})}{-3\cancel{\omega_0^2}} \;=\; -\frac{4}{3\sqrt{17}} \qquad\Rightarrow\qquad \phi \;=\; \tan^{-1}\left(-\frac{4}{3\sqrt{17}}\right) \;\sim\; \underline{\underline{2.83\,\mathrm{rad}}},$$

and the displacement of m is within ~ 18° of being out of phase with the ground.

e) Following the hint, we let $\omega^2 = \Omega$ in the second of equations (3), and examine,

$$\begin{aligned}\frac{dA^2}{d\Omega} \;&=\; \frac{d}{d\Omega}\left(\frac{\Omega^2\zeta_0^2}{\left(\omega_0^2-\Omega\right)^2 + 4\Omega\gamma^2}\right)\\ &=\; \frac{2\Omega\zeta_0^2\left[\left(\omega_0^2-\Omega\right)^2+4\Omega\gamma^2\right] - \left[2\left(\omega_0^2-\Omega\right)(-1)+4\gamma^2\right]\left(\Omega^2\zeta_0^2\right)}{\left[\;\cdots\;\right]^2} \;=\; 0,\end{aligned}$$

where we don't really care about the denominator since we'll be cross multiplying it with the zero. Thus,

$$\cancel{2\Omega\zeta_0^2}\left(\omega_0^4 - 2\omega_0^2\Omega + \Omega^2 + 4\Omega\gamma^2\right) + \cancel{2\Omega\zeta_0^2}\left(\omega_0^2\Omega - \Omega^2 - 2\gamma^2\Omega\right) \;=\; 0$$

$$\Rightarrow\quad \omega_0^4 - \omega_0^2\Omega + 2\Omega\gamma^2 \;=\; 0 \quad\Rightarrow\quad \Omega \;=\; \omega_{\mathrm{r}}^2 \;=\; \frac{\omega_0^4}{\omega_0^2 - 2\gamma^2}$$

$$\Rightarrow\quad \boxed{\omega_{\mathrm{r}} \;=\; \frac{\omega_0^2}{\sqrt{\omega_0^2-2\gamma^2}},} \tag{5}$$

as desired.

Substituting equation (5) into equation (3), we find,

$$\begin{aligned}A_{\max} \;&=\; \frac{\omega_0^4}{\omega_0^2-2\gamma^2}\,\frac{\zeta_0}{\sqrt{\left(\omega_0^2 - \frac{\omega_0^4}{\omega_0^2-2\gamma^2}\right)^2 + 4\frac{\omega_0^4}{\omega_0^2-2\gamma^2}\gamma^2}}\\ &=\; \frac{\omega_0^{\overset{2}{\cancel{4}}}}{\cancel{\omega_0^2-2\gamma^2}}\,\frac{\zeta_0}{\frac{\cancel{\omega_0^2}}{\cancel{\omega_0^2-2\gamma^2}}\sqrt{\left(\cancel{\omega_0^2} - 2\gamma^2 - \cancel{\omega_0^2}\right)^2 + 4\gamma^2\left(\omega_0^2-2\gamma^2\right)}} \;=\; \frac{\omega_0^2\zeta_0}{\sqrt{4\gamma^2+4\gamma^2\omega_0^2-8\gamma^2}}\end{aligned}$$

$$\Rightarrow\quad \boxed{A_{\max} \;=\; \frac{\omega_0^2\zeta_0}{2\gamma\sqrt{\omega_0^2-\gamma^2}},} \tag{6}$$

as desired.

f) Substituting $\gamma = \omega_0/\sqrt{17}$ (equation 4) into each of equations (5) and (6), we get:

$$\frac{\omega_\mathrm{r}}{\omega_0} = \frac{\cancel{\omega_0}}{\sqrt{\cancel{\omega_0^2} - 2\frac{\cancel{\omega_0^2}}{17}}} = \sqrt{\frac{17}{15}} \sim \underline{\underline{1.065}}$$

$$\frac{A_\mathrm{max}}{\zeta_0} = \frac{\cancel{\omega_0^2}}{2\frac{\cancel{\omega_0}}{\sqrt{17}}\sqrt{\cancel{\omega_0^2} - \frac{\cancel{\omega_0^2}}{17}}} = \frac{\sqrt{17}}{2\sqrt{1 - \frac{1}{17}}} = \underline{\underline{\frac{17}{8} = 2.125.}}$$

Thus, the resonant frequency is ∼6.5% higher than the natural oscillation frequency of the undamped spring, and the resonant amplitude is $2\frac{1}{8}$ times the oscillation amplitude of the ground.

Finally, substituting equations (4) and (5) into the first of equations (3), we get:

$$\tan\phi_\mathrm{r} = \frac{2\omega_\mathrm{r}\gamma}{\omega_0^2 - \omega_\mathrm{r}^2} = 2\,\frac{\omega_0^2}{\sqrt{\omega_0^2 - 2\frac{\omega_0^2}{17}}}\,\frac{\omega_0}{\sqrt{17}}\,\frac{1}{\omega_0^2 - \frac{\omega_0^4}{\omega_0^2 - 2\omega_0^2/17}}.$$

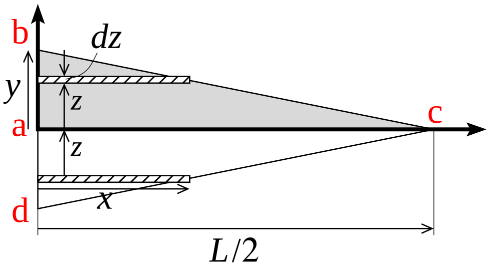
All factors of ω_0 cancel out, and we're left with:

$$\tan\phi_\mathrm{r} = \cancel{2}\sqrt{\frac{\cancel{17}}{15}}\,\frac{1}{\sqrt{\cancel{17}}}\left(-\frac{15}{\cancel{2}}\right) = -\sqrt{15} \quad\Rightarrow\quad \phi_\mathrm{r} = \tan^{-1}\left(-\sqrt{15}\right) \sim \underline{\underline{1.823\,\mathrm{rad}}}.$$

Thus, at resonant frequency, the displacement of m is ∼104° out of phase with the ground. Note that as $\gamma \to 0$, this phase difference → 90°.

Problem 24.

Solution: a) The potential energy of the water with the water level inclined is the same as though the mass in the triangle acd in the inset were raised to triangle abc. To do this, each horizontal strip of water of mass dm, length x, and thickness dz is raised a distance $2z$, where $0 \le z \le y$. Thus,



$$dU = 2gz\,dm \quad\Rightarrow\quad U = 2\rho wg\int_0^y xz\,dz,$$

since $dm = \rho xw\,dz$. Then, from similar triangles,

$$\frac{x}{y-z} = \frac{L/2}{y} \quad\Rightarrow\quad x = \frac{L(y-z)}{2y},$$

and we have:

$$U = \frac{\rho Lwg}{y}\int_0^y z(y-z)dz = \frac{\rho Lwg}{y}\left(\frac{yz^2}{2} - \frac{z^3}{3}\right)\Bigg|_0^y \quad\Rightarrow\quad \boxed{U(y) = \frac{\rho Lwg}{6}y^2,}$$

as desired. The fact that $U \propto y^2$ already tells us the system is a simple harmonic oscillator.

b) Given the result from part d, the total mechanical energy is:

$$E = U + K = \frac{\rho L w g}{6} y^2 + \frac{\rho w L^3}{60h} \dot{y}^2,$$

which has the classical form for the mechanical energy of a simple harmonic oscillator. Since ω_0^2 is given by the ratio of the coefficients, we have:

$$\omega_0^2 = \frac{\rho L w g}{6} \frac{60h}{\rho w L^3} = \frac{10gh}{L^2}$$

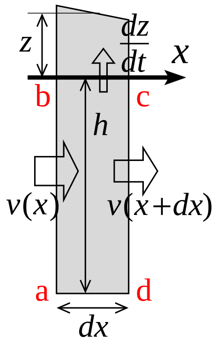
$$\Rightarrow \quad \boxed{\omega_0 = \frac{\sqrt{10gh}}{L}} \quad \text{and} \quad \boxed{T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi L}{\sqrt{10gh}}.}$$

For $L = 1.5\,\text{m}$ and $h = 0.3\,\text{m}$, $T_0 \sim 1.7\,\text{s}$, which seems about right given my distant memories of flooding the bathroom floor much to my mother's great consternation!

c) As for the mother of all bath tubs, the natural oscillation period of the Bay of Fundy, with $L = 5.0 \times 10^5\,\text{m}$ and an effective depth of $h = 50\,\text{m}$, is $T_0 \sim 4.49 \times 10^4\,\text{s} \sim 12.47\,\text{hr}$, which is very close to half the time it takes for the moon to return to the same position in the sky (24.9 hr).

Thus, the high tides in the Bay of Fundy can be explained as a resonance phenomenon.

d) (Bonus 5 points) For the kinetic energy, consider the inset, which depicts a narrow column of fluid of width dx and depth h below the level line (x-axis) and a trapezoid of average height z above the level line representing the rest of the water column up to the surface. As shown, the horizontal water speed at the left is given by $v(x)$, at the right $v(x + dx)$, and at the level surface, dz/dt, the rate at which the surface is either rising ($dz/dt > 0$) or falling ($dz/dt < 0$) at that place and moment.



In the fixed rectangle abcd of height h and width dx, as much water enters abcd in a given time, dt, as leaves (large arrows), and we write:

$$\cancel{\rho w} h v(x)\, \cancel{dt} - \cancel{\rho w} h v(x + dx)\, \cancel{dt} - \cancel{\rho w}\, dx \frac{dz}{dt} \cancel{dt} = 0$$

$$\Rightarrow \quad -\frac{v(x + dx) - v(x)}{dx} = \frac{1}{h}\frac{dz}{dt} \quad \Rightarrow \quad \frac{dv}{dx} = -\frac{1}{h}\dot{z}. \tag{1}$$

Again from similar triangles,

$$\frac{y - z}{x} = \frac{2y}{L} \quad \Rightarrow \quad z = y\left(1 - \frac{2x}{L}\right) \quad \Rightarrow \quad \dot{z} = \dot{y}\left(1 - \frac{2x}{L}\right),$$

since here, x is a fixed location and treated as a constant in the differentiation. Substituting this into equation (1), we get:

$$\frac{dv}{dx} = -\frac{\dot{y}}{h}\left(1-\frac{2x}{L}\right)$$

$$\Rightarrow \quad v(x) = \int \frac{dv}{dx}\,dx = -\frac{\dot{y}}{h}\int\left(1-\frac{2x}{L}\right)dx = -\frac{\dot{y}}{h}\left(x-\frac{x^2}{L}+c\right),$$

where c is a constant of integration. This is evaluated by imposing the boundary condition that at $x = 0$ (left end of the tub), the horizontal water speed is zero:

$$v(0) = -\frac{\dot{y}}{h}c = 0 \quad \Rightarrow \quad c = 0.$$

Thus,

$$v(x) = \frac{\dot{y}}{h}\left(\frac{x^2}{L}-x\right). \tag{2}$$

Note that this also satisfies the right boundary condition, namely $v(L) = 0$.

We are finally in a position to evaluate the kinetic energy:

$$dK = \frac{1}{2}v(x)^2 dm = \frac{\rho w h\,dx}{2}v(x)^2, \tag{3}$$

where we have ignored the portion of the trapezoid above the level line, as this is compensated by an equal amount of missing mass at the corresponding point in the right half of the tub. Thus, substituting equation (2) into (3), we find:

$$\begin{aligned} K &= \int_0^L dK = \frac{\rho w h}{2}\left(\frac{\dot{y}}{h}\right)^2\int_0^L\left(\frac{x^2}{L}-x\right)^2 dx \\ &= \frac{\rho w}{2h}\dot{y}^2\left(\frac{x^5}{5L^2}-\frac{x^4}{2L}+\frac{x^3}{3}\right)\Bigg|_0^L = \frac{\rho w}{2h}\dot{y}^2\frac{L^3}{30} \end{aligned}$$

$$\Rightarrow \quad \boxed{K(\dot{y}) = \frac{\rho w L^3}{60h}\dot{y}^2,}$$

as desired.

Problem 25.

Solution: In Cartesian coordinates,

$$\nabla\times\vec{F} = (\partial_y F_z - \partial_z F_y,\, \partial_z F_x - \partial_x F_z,\, \partial_x F_y - \partial_y F_x) = 0 \ \text{ for } \vec{F} \text{ conservative.}$$

a) Thus, for $\vec{F} = (xy, cx^2, z^3)$,

$$\nabla \times \vec{F} \;=\; (0, 0, 2cx - x) \;=\; 0 \quad \text{for} \quad \boxed{c \;=\; \tfrac{1}{2}.}$$

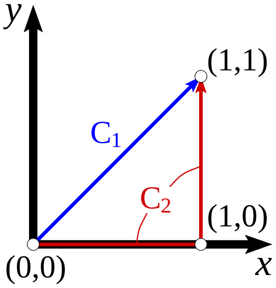
b) For $\vec{F} = \left(\dfrac{z}{y}, \dfrac{cxz}{y^2}, \dfrac{x}{y}\right)$,

$$\nabla \times \vec{F} \;=\; \left(-\frac{x}{y^2} - \frac{cx}{y^2}, \frac{1}{y} - \frac{1}{y}, \frac{cz}{y^2} + \frac{z}{y^2}\right) \;=\; \frac{1+c}{y^2}(-x, 0, z) \;=\; 0 \quad \text{for} \quad \boxed{c \;=\; -1.}$$

Problem 26.

Solution: For $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ and $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$,

$$\int_{\mathrm{C}} \vec{F}\cdot d\vec{r} = \int_{\mathrm{C}} \big(F_x dx + F_y dy\big). \tag{1}$$



a) Consider first $\vec{F} = x\hat{\imath} + y\hat{\jmath}$. For path C_1, $y = x$, $dy = dx$, and equation (1) becomes:

$$\int_{\mathrm{C}_1} \vec{F}\cdot d\vec{r} \;=\; 2\int_0^1 x\,dx \;=\; x^2\Big|_0^1 \;=\; 1.$$

For path C_2, we break it into two segments, the first between $(0, 0)$ and $(1, 0)$ where $dy = 0$, and the second between $(1, 0)$ and $(1, 1)$ where $dx = 0$. Thus, equation (1) becomes:

$$\int_{\mathrm{C}_2} \vec{F}\cdot d\vec{r} = \int_{\mathrm{C}_2} \big(x\,dx + y dy\big) \;=\; \int_0^1 x\,dx\Big|_{y=0} + \int_0^1 y\,dy\Big|_{x=1} \;=\; 1 \;=\; \int_{C_1} \vec{F}\cdot d\vec{r},$$

confirming that $\vec{F}$ is conservative.

b) Next, consider $\vec{F} = y\hat{\imath} - x\hat{\jmath}$. For path C_1, $y = x$, $dy = dx$, and equation (1) becomes:

$$\int_{\mathrm{C}_1} \vec{F}\cdot d\vec{r} \;=\; \int_0^1 \big(y\,dx - x\,dy\big) \;=\; \int_0^1 (x - x) dy \;=\; 0.$$

Conversely, along the two segments of path C_2, equation (1) becomes:

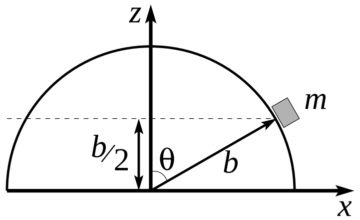
$$\int_{\mathrm{C}_2} \vec{F}\cdot d\vec{r} = \int_{\mathrm{C}_2} \big(y\,dx - x\,dy\big) \;=\; \int_0^1 y\,dx\Big|_{y=0} - \int_0^1 x\,dy\Big|_{x=1} \;=\; -1 \;\neq\; \int_{C_1} \vec{F}\cdot d\vec{r},$$

and $\vec{F}$ as given in this part is *not* conservative.

Problem 27.

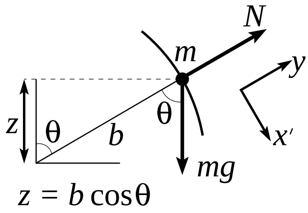
Solution: This problem is very similar to example 4.6.1 (eds. 6 and 7), with m starting at a different point.

Conserving E, we get,



$$E = mg\frac{b}{2} = mgz + \tfrac{1}{2}mv^2$$
$$\Rightarrow \quad v^2 = g(b-2z).$$

Then, from the FBD, we have in the y' direction:



$$\begin{aligned} -mg\cos\theta + N &= -ma_y = -\frac{mv^2}{b} = -\frac{mg(b-2z)}{b} \\ &= -\frac{mg(b-2b\cos\theta)}{b} = -mg + 2mg\cos\theta \\ \Rightarrow \quad 3mg\cos\theta &= mg + N. \end{aligned}$$

The mass, m, leaves the surface when $N = 0$,

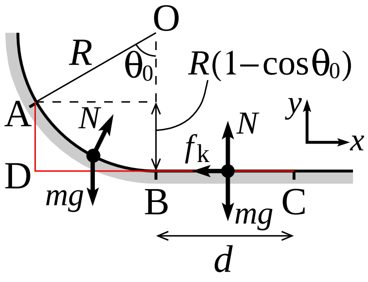
$$\Rightarrow \quad \cos\theta = \frac{1}{3} = \frac{z}{b} \quad \Rightarrow \quad \boxed{z = \frac{b}{3}.}$$

Problem 28.

Solution: As indicated in the diagram, gravity, mg, and a normal force, N, act on m along the entire track, whereas the kinetic friction force,

$$f_\mathrm{k} = \mu_\mathrm{k}N = \mu_\mathrm{k}mg = \text{constant}, \tag{1}$$

acts on m only along BC in the $-x$ direction.



Since gravity is conservative, W_{mg} is path independent and we need not evaluate it along the path m actually takes (ABC). Instead, consider the red path ADC with a vertical drop AD of height $h = R(1-\cos\theta_0)$ followed by a horizontal translation DC. Along this path, mg does work only along AD, which is much easier to compute than over ABC:

$$W_{mg} = \int_h^0 m\vec{g}\cdot d\vec{y} = -\int_h^0 mg\,dy = -mg(0-h) = mgR(1-\cos\theta_0). \tag{2}$$

Since $W_{mg} > 0$, gravity acts to accelerate m.

As always, the normal force, $\vec{N}$, does no work leaving f_k which is non-zero only along BC. Thus, work done by friction is:

$$W_{f_\text{k}} \;=\; \int_{x_\text{B}}^{x_\text{C}} \vec{f}_\text{k}\cdot d\vec{x} \;=\; -\int_{x_\text{B}}^{x_\text{C}} f_\text{k} dx \;=\; -\mu_\text{k} mg(x_\text{C}-x_\text{B}) \;=\; -\mu_\text{k} mgd, \tag{3}$$

using equation (1). Since $W_{f_\text{k}} < 0$, friction acts to decelerate m.

Thus, from the W-K theorem and equations (2) and (3), we can write:

$$\sum W \;=\; W_{mg} + \cancelto{0}{W_N} + W_{f_\text{k}} \;=\; mgR(1-\cos\theta_0) - \mu_\text{k} mgd \;=\; \Delta K \;=\; K_\text{C} - K_\text{A} \;=\; 0$$

$$\Rightarrow \quad \boxed{d \;=\; \frac{R(1-\cos\theta_0)}{\mu_\text{k}}.}$$

Problem 29.

Solution: We start with equation 4.3.9 (in ed. 7) giving the trajectory of a projectile in the absence of air resistance, and generalised for a non-zero starting height, z_0:

$$z(x) - z_0 \;=\; x\tan\alpha - \frac{g}{2v_0^2\cos^2\!\alpha}x^2. \tag{1}$$

For $z_0 = h$, the range, R, is the value of x when $z = 0$ (see figure). Thus, for a given α,

$$z(R) \;=\; 0 \;=\; h + R\tan\alpha - \frac{g}{2v_0^2\cos^2\!\alpha}R^2$$

$$\Rightarrow \quad R^2 - \frac{2v_0^2R}{g}\tan\alpha\cos^2\!\alpha - \frac{2v_0^2h}{g}\cos^2\!\alpha,$$

a quadratic in R. Since $2\tan\alpha\cos^2\!\alpha = 2\sin\alpha\cos\alpha = \sin 2\alpha$ and $2\cos^2\!\alpha = \cos 2\alpha + 1$,

$$R^2 - R\frac{v_0^2}{g}\sin 2\alpha - \frac{v_0^2h}{g}(\cos 2\alpha + 1) \;=\; 0, \tag{2}$$

To find the angle, α, that extremises R, we need to set $dR/d\alpha = 0$. To do that, we *could* first solve Eq. (2) for R using the quadratic formula:

$$R \;=\; \frac{v_0^2\sin 2\alpha}{2g} \pm \sqrt{\frac{v_0^4\sin^2\!2\alpha}{4g^2} + \frac{v_0^2h}{g}(\cos 2\alpha + 1)},$$

and *then* find $dR/d\alpha$, but this looks awful!! So instead, let's differentiate Eq. (2) *implicitly*:

$$2R\cancelto{0}{\frac{dR}{d\alpha}} - \cancelto{0}{\frac{dR}{d\alpha}}\frac{v_0^2}{g}\sin 2\alpha - R\frac{v_0^2}{g}2\cos 2\alpha + \frac{v_0^2h}{g}2\sin 2\alpha \;=\; 0$$

$$\Rightarrow \quad -R\cos 2\alpha + h\sin 2\alpha \;=\; 0 \quad \Rightarrow \quad R \;=\; h\tan 2\alpha. \tag{3}$$

Substituting equation (3) into equation (2), we get:

$$h^{\cancel{2}}\tan^2 2\alpha - \cancel{h}\tan 2\alpha\frac{v_0^2}{g}\sin 2\alpha - \frac{v_0^2\cancel{h}}{g}(\cos 2\alpha + 1) \;=\; 0$$

$$\Rightarrow \quad h\frac{\sin^2 2\alpha}{\cos^2 2\alpha} - \frac{v_0^2}{g}\frac{\sin^2 2\alpha}{\cos 2\alpha} - \frac{v_0^2}{g}(\cos 2\alpha + 1) \;=\; 0,$$

eliminating the tan function. Next, putting terms proportional to v_0^2/g on the RHS, we get,

$$h\frac{\sin^2 2\alpha}{\cos^2 2\alpha} \;=\; \frac{v_0^2}{g}\left(\frac{\sin^2 2\alpha}{\cos 2\alpha} + \cos 2\alpha + 1\right) \;=\; \frac{v_0^2}{g}\,\frac{\sin^2 2\alpha + \cos^2 2\alpha + \cos 2\alpha}{\cos 2\alpha}$$

$$\Rightarrow \quad h\frac{\sin^2 2\alpha}{\cos^{\cancel{2}} 2\alpha} \;=\; \frac{v_0^2}{g}\,\frac{1+\cos 2\alpha}{\cancel{\cos 2\alpha}}.$$

Then, use identities: $\sin 2\alpha = 2\sin\alpha\cos\alpha$; $\cos 2\alpha = 1 - \sin^2\alpha$; $1 + \cos 2\alpha = \cos^2\alpha$, to get:

$$h\frac{\overset{2}{\cancel{4}}\sin^2\alpha\,\cancel{\cos^2\alpha}}{1 - 2\sin^2\alpha} \;=\; \frac{v_0^2}{g}\cancel{2}\cancel{\cos^2\alpha} \quad \Rightarrow \quad \frac{gh}{v_0^2} \;=\; \frac{1 - 2\sin^2\alpha}{2\sin^2\alpha} \;=\; \frac{1}{2\sin^2\alpha} - 1$$

$$\Rightarrow \quad \frac{1}{2\sin^2\alpha} \;=\; 1 + \frac{gh}{v_0^2} \;=\; \frac{v_0^2 + gh}{v_0^2} \quad \Rightarrow \quad \boxed{\sin^2\alpha \;=\; \frac{1}{2}\,\frac{v_0^2}{v_0^2 + gh},} \tag{4}$$

as desired.

b) To find R, we first note from Eq. (4) that:

$$\cos 2\alpha \;=\; 1 - 2\sin^2\alpha \;=\; 1 - \frac{v_0^2}{v_0^2 + gh} \;=\; \frac{gh}{v_0^2 + gh}.$$

Then, from Eq. (3), we have:

$$R_{\max} \;=\; h\tan 2\alpha \;=\; h\frac{\sin 2\alpha}{\cos 2\alpha} \;=\; h\frac{\sqrt{1 - \cos^2 2\alpha}}{\cos 2\alpha} \;=\; \cancel{h}\sqrt{1 - \left(\frac{gh}{v_0^2 + gh}\right)^2}\,\frac{v_0^2 + gh}{g\cancel{h}}$$

$$= \; \frac{1}{g}\sqrt{(v_0^2 + gh)^2 - (gh)^2} \;=\; \frac{1}{g}\sqrt{v_0^4 + 2v_0^2 gh}$$

$$\Rightarrow \quad \boxed{R_{\max} \;=\; \frac{v_0^2}{g}\sqrt{1 + \frac{2gh}{v_0^2}}.}$$

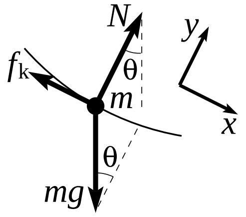
Note that for $h = 0$, this reduces further to $R_{\max} = v_0^2/g$, the range worked out in PHYS 1210 for a cannon on the same level as the target.

Problem 30.

Solution: a) Consider the FBD for m when it is somewhere between points A and B, and thus $0 < \theta < \theta_0$. From Newton's 2^{nd} Law, we have:

$$x/ \qquad -f_{\text{k}} + mg\sin\theta \;=\; ma_x \;=\; ma_{\text{tan}}; \qquad (3)$$

$$y/ \qquad N - mg\cos\theta \;=\; ma_y \;=\; m\frac{v^2}{R} \;=\; ma_{\text{cp}}, \qquad (4)$$



where, since path AB is circular, a_x is a tangential acceleration, and a_y is a centripetal acceleration. Then, from the hint,

$$a_{\text{tan}} \;=\; -\frac{1}{2}\frac{da_{\text{cp}}}{d\theta}.$$

Substituting this along with $f_{\text{k}} = \mu_{\text{k}}N$ into equation (3) yields:

$$-\mu_{\text{k}}N + mg\sin\theta \;=\; -m\frac{1}{2}\frac{da_{\text{cp}}}{d\theta} \quad\Rightarrow\quad N \;=\; \frac{m}{\mu_{\text{k}}}\left(g\sin\theta + \frac{1}{2}\frac{da_{\text{cp}}}{d\theta}\right),$$

and substituting this into equation (4) gives us a first order differential equation for a_{cp}:

$$\frac{\cancel{m}}{\mu_{\text{k}}}\left(g\sin\theta + \frac{1}{2}\frac{da_{\text{cp}}}{d\theta}\right) - \cancel{m}g\cos\theta \;=\; \cancel{m}a_{\text{cp}}$$

$$\Rightarrow \quad \boxed{\frac{da_{\text{cp}}}{d\theta} - 2\mu_{\text{k}}a_{\text{cp}} \;=\; 2g(\mu_{\text{k}}\cos\theta - \sin\theta),}$$

in agreement with equation (1).

b) To solve equation (1), first consider the *homogeneous* equation:

$$\frac{da_{\text{cp,h}}}{d\theta} - 2\mu_{\text{k}}a_{\text{cp,h}} \;=\; 0,$$

whose solution we can write down by inspection:

$$a_{\text{cp,h}} \;=\; \alpha e^{2\mu_{\text{k}}\theta},$$

where α is a constant of integration. Alternately, one could recognise the homogeneous equation as a first order separable ODE, and solve directly to get the same result.

Next, we seek a particular solution to equation (1). Again by inspection, we see that since the RHS has both a cosine and a sine term, a particular solution must have the form:

$$a_{\text{cp,p}} \;=\; \beta\cos\theta + \gamma\sin\theta,$$

which we substitute into equation (1) directly to solve for β and γ. Thus,

$$-\beta\sin\theta + \gamma\cos\theta - 2\mu_{\text{k}}(\beta\cos\theta + \gamma\sin\theta) \;=\; 2g(\mu_{\text{k}}\cos\theta - \sin\theta)$$

$$\Rightarrow \quad (-\beta - 2\mu_\mathrm{k}\gamma + 2g)\sin\theta \;=\; (2g\mu_\mathrm{k} - \gamma + 2\mu_\mathrm{k}\beta)\cos\theta.$$

Since $\cos\theta$ and $\sin\theta$ are linearly independent functions, the coefficients must be zero:

$$\left.\begin{aligned} -\beta - 2\mu_\mathrm{k}\gamma + 2g \;&=\; 0 \\ 2g\mu_\mathrm{k} - \gamma + 2\mu_\mathrm{k}\beta \;&=\; 0 \end{aligned}\right\} \quad\Rightarrow\quad \beta \;=\; 2g\frac{1-2\mu_\mathrm{k}^2}{1+4\mu_\mathrm{k}^2} \quad\text{and}\quad \gamma \;=\; 6g\frac{\mu_\mathrm{k}}{1+4\mu_\mathrm{k}^2}. \tag{5}$$

Thus, the solution to equation (1) is:

$$a_\mathrm{cp} \;=\; a_\mathrm{cp,h} + a_\mathrm{cp,p} \;=\; \alpha e^{2\mu_\mathrm{k}\theta} + \beta\cos\theta + \gamma\sin\theta. \tag{6}$$

To evaluate α, we apply initial conditions: at $t=0$, $\theta=\theta_0$ and $a_\mathrm{cp}=0$ $(v=0)$. Thus,

$$0 \;=\; \alpha e^{2\mu_\mathrm{k}\theta_0} + \beta\cos\theta_0 + \gamma\sin\theta_0 \quad\Rightarrow\quad \alpha \;=\; -(\beta\cos\theta_0 + \gamma\sin\theta_0)e^{-2\mu_\mathrm{k}\theta_0}. \tag{7}$$

Substituting equation (7) into (6), we get:

$$a_\mathrm{cp} \;=\; -(\beta\cos\theta_0 + \gamma\sin\theta_0)e^{2\mu_\mathrm{k}(\theta-\theta_0)} + \beta\cos\theta + \gamma\sin\theta. \tag{8}$$

Now, at point B, $\theta=0$ and equation (8) becomes:

$$a_\mathrm{cp,B} \;=\; \frac{v_\mathrm{B}^2}{R} \;=\; -(\beta\cos\theta_0 + \gamma\sin\theta_0)e^{-2\mu_\mathrm{k}\theta_0} + \beta$$

$$\Rightarrow \quad \boxed{v_\mathrm{B}^2 \;=\; R\Big[\beta\big(1-\cos\theta_0 e^{-2\mu_\mathrm{k}\theta_0}\big) - \gamma\sin\theta_0 e^{-2\mu_\mathrm{k}\theta_0}\Big]},$$

which agrees with equation (2) after equations (5) have been substituted in.

c) To find the distance d that m slides beyond point B and comes to rest at point C, we use the W-K theorem. Between points B and C, the only force that does work is $f_\mathrm{k} = \mu_\mathrm{k} mg$, and thus,

$$\sum W \;=\; \Delta K \quad\Rightarrow\quad -\mu_\mathrm{k} mgd \;=\; \cancelto{0}{K_\mathrm{C}} - K_\mathrm{B} \;=\; -\tfrac{1}{2}mv_\mathrm{B}^2$$

$$\Rightarrow \quad d \;=\; \frac{1}{2\mu_\mathrm{k} g}v_\mathrm{B}^2 \;=\; \frac{R}{2\mu_\mathrm{k} g}\Big[\beta\big(1-\cos\theta_0 e^{-2\mu_\mathrm{k}\theta_0}\big) - \gamma\sin\theta_0 e^{-2\mu_\mathrm{k}\theta_0}\Big]$$

$$\Rightarrow \quad \boxed{d \;=\; \frac{R}{\mu_\mathrm{k}(1+4\mu_\mathrm{k}^2)}\Big[\big(1-2\mu_\mathrm{k}^2\big)\big(1-\cos\theta_0 e^{-2\mu_\mathrm{k}\theta_0}\big) - 3\mu_\mathrm{k}\sin\theta_0 e^{-2\mu_\mathrm{k}\theta_0}\Big]},$$

using equations (5).

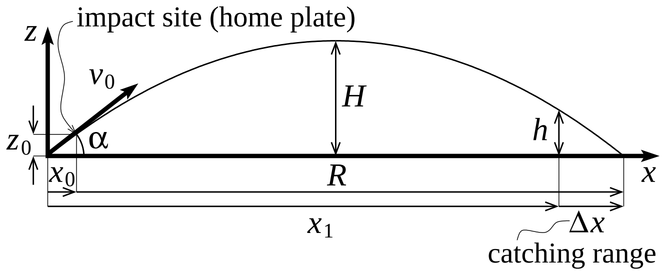
Problem 31.

Solution: a) The figure depicts the ball's trajectory in the suggested coordinate system, where the ball is struck at $(x,z) = (x_0, z_0)$. With the trajectory beginning at the origin, we have from equations (4.3.9), (4.3.11), and (4.3.14, corrected) in ed. 7:

$$z \;=\; x\tan\alpha - \frac{g}{2v_0^2\cos^2\!\alpha}\,x^2; \quad (1)$$

$$H \;=\; \frac{v_0^2\sin^2\!\alpha}{2g} \text{ (max. height)}; \quad (2)$$

$$R + x_0 \;=\; \frac{v_0^2\sin 2\alpha}{g} \text{ (range).} \qquad (3)$$



Note that v_0 and α are the initial velocity and elevation angle the ball would have if launched from the origin rather than by the batter. Note further that the total range in (3) is the batter's range, R, plus the distance between the origin and home plate, x_0.

Solving equation (2) for v_0^2, we get:

$$v_0^2 \;=\; \frac{2gH}{\sin^2\!\alpha}. \tag{4}$$

Substituting this into equation (1) gives us:

$$z \;=\; x\tan\alpha - \frac{\cancel{g}\sin^2\!\alpha}{4\cancel{g}H\cos^2\!\alpha}x^2 \;=\; x\tan\alpha - \frac{1}{4H}(x\tan\alpha)^2, \tag{5}$$

which, at $x = x_0$ (location of home plate), gives us:

$$z_0 \;=\; x_0\tan\alpha - \frac{1}{4H}(x_0\tan\alpha)^2 \quad\Rightarrow\quad (x_0\tan\alpha)^2 - 4H(x_0\tan\alpha) + 4Hz_0 \;=\; 0$$

$$\Rightarrow\quad x_0\tan\alpha \;=\; \frac{4H \pm \sqrt{16H^2 - 16Hz_0}}{2} \;=\; 2H\Big(1 - \sqrt{1 - z_0/H}\Big), \tag{6}$$

choosing the smaller of the two roots ($-$) and thus closer to the origin.

Next, substitute equation (4) into (3) to get:

$$R + x_0 \;=\; \frac{2\cancel{g}H}{\sin^{\cancel{2}}\!\alpha}\,\frac{2\cancel{\sin\alpha}\cos\alpha}{\cancel{g}} \;=\; \frac{4H}{\tan\alpha} \quad\Rightarrow\quad x_0\tan\alpha \;=\; 4H - R\tan\alpha. \tag{7}$$

Then, comparing equations (6) and (7), we get:

$$4H - R\tan\alpha \;=\; 2H\Big(1 - \sqrt{1 - z_0/H}\Big) \quad\Rightarrow\quad \tan\alpha \;=\; \frac{2H\big(1 + \sqrt{1 - z_0/H}\big)}{R}.$$

Now, noting that $H = 5.07 = 3\big(\frac{13}{10}\big)^2$, $z_0 = \frac{3}{4}$, and $R = 25$,

$$1 - \frac{z_0}{H} \;=\; 1 - \frac{\cancel{3}/4}{\cancel{3}\,\big(\frac{13}{10}\big)^2} \;=\; 1 - \frac{25}{169} \;=\; \frac{144}{169} \quad\Rightarrow\quad \sqrt{1 - \frac{z_0}{H}} \;=\; \frac{12}{13}$$

$$\Rightarrow\quad \tan\alpha \;=\; \frac{1}{25}\,6\bigg(\frac{13}{10}\bigg)^2\bigg(1 + \frac{12}{13}\bigg) \;=\; \frac{39}{50},$$

which, when substituted into equation (5) gives:

$$z = \frac{39}{50}x - \frac{1}{12}\left(\frac{10}{13}\right)^2\left(\frac{39}{50}\right)^2 x^2 = \frac{39}{50}x - \frac{3}{100}x^2 \quad \Rightarrow \quad \boxed{3x^2 - 78x + 100z = 0,} \tag{8}$$

as required.

b) The batter, located at $x = x_0$, strikes the ball from a height $z = z_0 = \frac{3}{4}$. Substituting this into equation (8) yields:

$$3x_0^2 - 78x_0 + 75 = 0,$$

which one can see by inspection admits $\boxed{x_0 = 1.00 \text{ m}}$ (or use quadratic formula).

c) Let x_1 be the horizontal distance of the ball from the origin when its height above ground is $z = h = 2.07$ m (see figure). Then, from equation (8),

$$3x_1^2 - 78x_1 + 207 = 0 \quad \Rightarrow \quad x_1^2 - 26x_1 + 69 = 0$$

$$\Rightarrow \quad x_1 = \frac{26 \pm \sqrt{676 - 276}}{2} = 23.0 \text{ m},$$

choosing the root furthest from home plate (+). Thus, the ball will be $x_1 - x_0 = 22.0$ m from home plate when its height is 2.07 m. Since the range of the batter is $R = 25.0$ m,

$$\boxed{\text{the outfielder needs to be between 22.0 and 25.0 m from home plate to catch the fly-ball.}}$$

As suggested by Michael Power (class of '18), the following may be a more intuitive approach for parts a) and b):

The general form for the trajectory starting at the origin is:

$$z = ax - bx^2, \tag{9}$$

and we know three points that lie on this trajectory:

$$(x_0, z_0); \qquad \big(\tfrac{1}{2}(R + x_0), H\big); \qquad \text{and} \qquad (R + x_0, 0).$$

Substituting each into (9) gives us:

$$z_0 = ax_0 - bx_0^2; \tag{10}$$

$$H = \frac{a}{2}(R + x_0) - \frac{b}{4}(R + x_0)^2; \quad \text{and} \tag{11}$$

$$0 = a(R + x_0) - b(R + x_0)^2, \tag{12}$$

which may be solved for a, b, and x_0. To this end, start with (12) to find:

$$a = b(R + x_0), \tag{13}$$

and substitute this into (10) to find:

$$z_0 \;=\; b(R+x_0)x_0 - bx_0^2 \;=\; bRx_0 \quad\Rightarrow\quad b \;=\; \frac{z_0}{Rx_0} \tag{14}$$

$$\Rightarrow\quad a \;=\; \frac{z_0}{Rx_0}(R+x_0). \tag{15}$$

Then, substituting each of equations (15) and (14) into (11) gives us:

$$H \;=\; \frac{z_0}{2Rx_0}(R+x_0)^2 - \frac{z_0}{4Rx_0}(R+x_0)^2 \;=\; \frac{z_0}{4Rx_0}(R+x_0)^2$$

$$\Rightarrow\quad 4HRx_0 \;=\; z_0(R^2+2Rx_0+x_0^2) \quad\Rightarrow\quad x_0^2 + 2R\left(1-\frac{2H}{z_0}\right)x_0 + R^2 \;=\; 0.$$

Putting in numbers ($R = 25$, $H = 3(169/100)$ and $z_0 = 3/4$), we get:

$$x_0^2 + 50\left(1 - 2(\cancel{3})\frac{169}{100}\frac{4}{\cancel{3}}\right)x_0 + 625 \;=\; 0$$

$$\Rightarrow\quad x_0^2 - 626x_0 + 625 \;=\; 0 \quad\Rightarrow\quad x_0 \;=\; 1 \quad\text{or}\quad 625,$$

the latter dismissed as being non-physical (beyond R). Thus, $x_0 = 1$ (distance between origin and home plate; part b), and equations (14) and (13) give us:

$$b \;=\; \frac{3/4}{(25)(1)} \;=\; \frac{3}{100} \qquad\text{and}\qquad a \;=\; \frac{3}{100}(26) \;=\; \frac{78}{100}.$$

Substituting these into equation (9) gives us our desired result for part a):

$$z \;=\; \frac{78}{100}x - \frac{3}{100}x^2 \quad\Rightarrow\quad \boxed{3x^2 - 78x + 100z \;=\; 0.}$$

Problem 32.

Solution: a) From equation (1), we can write out the x- and z-components separately:

$$x(t) \;=\; \frac{v_{0x}}{\gamma}\left(1-e^{-\gamma t}\right); \tag{3}$$

$$z(t) \;=\; \frac{1}{\gamma}\left(v_{0z}+\frac{g}{\gamma}\right)\left(1-e^{-\gamma t}\right) - \frac{gt}{\gamma}. \tag{4}$$

From equation (3), we have:

$$1-e^{-\gamma t} \;=\; \frac{\gamma x}{v_{0x}} \tag{5}$$

$$\Rightarrow\quad e^{-\gamma t} \;=\; 1-\frac{\gamma x}{v_{0x}} \quad\Rightarrow\quad t \;=\; -\frac{1}{\gamma}\ln\left(1-\frac{\gamma x}{v_{0x}}\right). \tag{6}$$

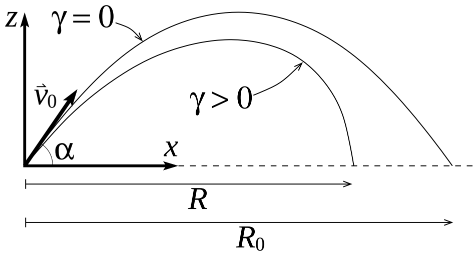
Substituting equation (5) into the first term of equation (4) and equation (6) into the second term, we get:

$$\boxed{z(x) \;=\; \frac{1}{\cancel{\gamma}}\left(v_{0z}+\frac{g}{\gamma}\right)\frac{\cancel{\gamma}x}{v_{0x}}+\frac{g}{\gamma^2}\ln\left(1-\frac{\gamma x}{v_{0x}}\right),}$$

as desired.

b) To find the range, R, as indicated in the figure, set $z(R)=0$ in equation (2). Thus,

$$\frac{R}{v_{0x}}\left(v_{0z}+\frac{g}{\gamma}\right)+\frac{g}{\gamma^2}\ln\left(1-\frac{\gamma R}{v_{0x}}\right) \;=\; 0. \qquad (7)$$



To make further progress, do a Maclaurin expansion on the logarithm:

$$\ln(1+u) \;=\; u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\ldots$$

Thus, for small γ ($\gamma \ll v_0/g$), equation (7) becomes:

$$\frac{R}{v_{0x}}\left(v_{0z}+\frac{g}{\gamma}\right)+\frac{g}{\gamma^2}\left(-\frac{\gamma R}{v_{0x}}-\frac{(\gamma R)^2}{2v_{0x}^2}-\frac{(\gamma R)^3}{3v_{0x}^3}-\ldots\right)$$

$$=\frac{v_{0z}}{v_{0x}}R+\cancel{\frac{Rg}{v_{0x}\gamma}}-\cancel{\frac{gR}{\gamma v_{0x}}}-\frac{gR^2}{2v_{0x}^2}-\frac{g\gamma R^3}{3v_{0x}^3}-\ldots \;=\; 0$$

$$\Rightarrow \quad v_{0z}-\frac{gR}{2v_{0x}}-\frac{g\gamma R^2}{3v_{0x}^2}-\mathcal{O}(\gamma^2) \;=\; 0,$$

having multiplied through by v_{0x}/R. This is as far as we got in class.

Next, drop terms of $\mathcal{O}(\gamma^2)$ to leave a quadratic in R:

$$R^2+\frac{3v_{0x}}{2\gamma}R-\frac{3v_{0x}^2v_{0z}}{g\gamma} \;\approx\; 0,$$

on which we use the quadratic formula to get:

$$R \;=\; -\frac{3v_{0x}}{4\gamma}\pm\sqrt{\frac{9v_{0x^2}}{16\gamma^2}+\frac{3v_{0x}^2v_{0z}}{g\gamma}} \;=\; -\frac{3v_{0x}}{4\gamma}+\frac{3v_{0x}}{4\gamma}\sqrt{1+\frac{16\gamma v_{0z}}{3g}},$$

keeping the + root so that $R>0$. Next, apply a binomial expansion to the radical to get:

$$R \;=\; -\cancel{\frac{3v_{0x}}{4\gamma}}+\frac{3v_{0x}}{4\gamma}\left(\cancel{1}+\frac{1}{2}\,\frac{16\gamma v_{0z}}{3g}-\frac{1}{8}\,\frac{256\gamma^2v_{0z}^2}{9g^2}+\mathcal{O}(\gamma^3)\right)$$

$$=\frac{2v_{0x}v_{0z}}{g}-\frac{8\gamma v_{0x}v_{0z}^2}{3g^2}+\mathcal{O}(\gamma^2) \;=\; \frac{2v_{0x}v_{0z}}{g}\left(1-\frac{4\gamma v_{0z}}{3g}+\mathcal{O}(\gamma^2)\right).$$

On substituting $v_{0x} = v_0 \cos\alpha$ and $v_{0z} = v_0 \sin\alpha$, we get:

$$R \;=\; \frac{2v_0^2 \sin\alpha\cos\alpha}{g}\left(1 - \frac{4\gamma v_0}{3g}\sin\alpha + \mathcal{O}(\gamma^2)\right) \;=\; \boxed{R_0\left(1 - \frac{4\gamma v_0}{3g}\sin\alpha + \mathcal{O}(\gamma^2)\right),}$$

as desired (having used the trig identity $2\sin\alpha\cos\alpha = \sin 2\alpha$).

Problem 33.

Solution: a) From equations 4.4.4 in the text, we have for a 2-D isotropic oscillator:

$$x(t) \;=\; B\cos(\omega t + \alpha) \quad\Rightarrow\quad \dot{x}(t) \;=\; -B\omega\sin(\omega t + \alpha); \tag{1}$$

$$y(t) \;=\; C\cos(\omega t + \beta) \quad\Rightarrow\quad \dot{y}(t) \;=\; -C\omega\sin(\omega t + \beta). \tag{2}$$

Using the initial conditions, equations (1) and (2) give:

$$B\cos\alpha \;=\; A; \quad -B\omega\sin\alpha \;=\; 0 \quad\Rightarrow\quad \alpha \;=\; 0; \quad B \;=\; A,$$

$$C\cos\beta \;=\; 4A; \quad -C\cancel{\omega}\sin\beta \;=\; 3\cancel{\omega}A \quad\Rightarrow\quad C^2\cos^2\beta + C^2\sin^2\beta \;=\; C^2 \;=\; 25A^2.$$

$$\Rightarrow\quad C \;=\; 5A \quad\Rightarrow\quad \cos\beta \;=\; \frac{4}{5}; \quad \sin\beta \;=\; -\frac{3}{5}.$$

Thus, the coordinates of the oscillator are given by:

$$x(t) \;=\; A\cos\omega t \tag{3}$$

$$\begin{aligned} y(t) \;&=\; C\cos(\omega t + \beta) \;=\; C(\cos\omega t\cos\beta - \sin\omega t\sin\beta) \\ &=\; 5A\left(\tfrac{4}{5}\cos\omega t + \tfrac{3}{5}\sin\omega t\right) \\ &=\; 4A\cos\omega t + 3A\sin\omega t. \end{aligned} \tag{4}$$

To find the trajectory (path taken by m), we eliminate t between (3) and (4). To this end, from (3), we have:

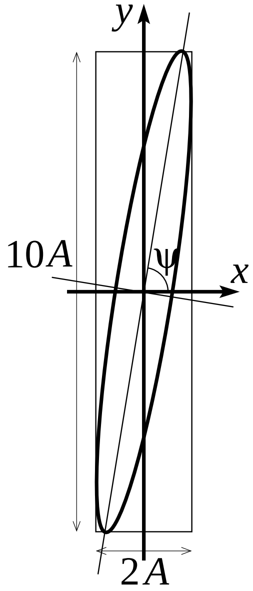
$$\cos\omega t \;=\; \frac{x}{A} \quad\Rightarrow\quad \sin\omega t \;=\; \sqrt{1 - \cos^2\omega t} \;=\; \sqrt{1 - \frac{x^2}{A^2}}.$$

We can therefore eliminate the trig functions from (4) and get:

$$y(x) \;=\; 4\cancel{A}\frac{x}{\cancel{A}} + 3A\sqrt{1 - \frac{x^2}{A^2}} \;=\; 4x + 3\sqrt{A^2 - x^2}$$

$$\Rightarrow\quad 3\sqrt{A^2 - x^2} \;=\; y - 4x \quad\Rightarrow\quad 9A^2 - 9x^2 \;=\; 16x^2 - 8xy + y^2$$

$$\Rightarrow\quad 25x^2 - 8xy + y^2 \;=\; 9A^2 \quad\Rightarrow\quad \boxed{\frac{x^2}{A^2} - 2xy\frac{4/5}{A(5A)} + \frac{y^2}{(5A)^2} \;=\; \frac{9}{25}.}$$



According to equation 4.4.10 and Figure 4.4.2 of eds. 6 and 7, this is an ellipse that fits inside a box of width $2\Delta x = 2A$ and a height $2\Delta y = 10A$.

b) According to equation 4.4.15 of eds. 6 and 7, the angle of inclination of the semi-major axis of the ellipse (relative to the $+x$ axis), ψ, is given by:

$$\tan 2\psi \;=\; \frac{2(A)(5A)}{A^2-(5A)^2}\frac{4}{5} \;=\; \frac{8}{-24} \;=\; -\frac{1}{3} \quad\Rightarrow\quad \boxed{\psi \;=\; \frac{1}{2}\tan^{-1}\!\left(-\frac{1}{3}\right) \;\sim\; 80.8^\circ.}$$

Problem 34.

Solution: a) According to equation 4.4.19 in the text, the motion of the non-isotropic oscillator is given by:

$$\begin{aligned} x(t) &= A\cos(\omega_x t+\alpha) &\Rightarrow\quad v_x(t) &= -A\omega_x\sin(\omega_x t+\alpha);\\ y(t) &= B\cos(\omega_y t+\beta) &\Rightarrow\quad v_y(t) &= -B\omega_y\sin(\omega_y t+\beta);\\ z(t) &= C\cos(\omega_z t+\gamma) &\Rightarrow\quad v_z(t) &= -C\omega_z\sin(\omega_z t+\gamma), \end{aligned}$$

where the frequencies are given by:

$$\omega_x \;=\; \sqrt{\frac{2k_x}{m}} \;=\; \pi; \qquad \omega_y \;=\; \sqrt{\frac{2k_y}{m}} \;=\; 2\pi; \qquad \omega_z \;=\; \sqrt{\frac{2k_z}{m}} \;=\; 3\pi,$$

and the phases α, β, and γ are determined from initial conditions:

$$\begin{aligned} \vec{r}(0) \;=\; \vec{0} \quad&\Rightarrow\quad A\cos\alpha \;=\; B\cos\beta \;=\; C\cos\gamma \;=\; 0 \quad\Rightarrow\quad \alpha \;=\; \beta \;=\; \gamma \;=\; \frac{\pi}{2},\\ \vec{v}(0) \;=\; \frac{1}{\sqrt{3}}(1,1,1) \quad&\Rightarrow\quad A\pi\sin\alpha \;=\; 2B\pi\sin\beta \;=\; 3C\pi\sin\gamma \;=\; -\frac{1}{\sqrt{3}}\\ &\Rightarrow\quad A \;=\; -\frac{1}{\sqrt{3}\,\pi}; \qquad B \;=\; -\frac{1}{2\sqrt{3}\,\pi}; \qquad C \;=\; -\frac{1}{3\sqrt{3}\,\pi}. \end{aligned}$$

Now, since $\cos(\theta+\pi/2) = -\sin\theta$, we have finally that:

$$\boxed{x(t) \;=\; \frac{1}{\sqrt{3}\,\pi}\sin(\pi t); \qquad y(t) \;=\; \frac{1}{2\sqrt{3}\,\pi}\sin(2\pi t); \qquad z(t) \;=\; \frac{1}{3\sqrt{3}\,\pi}\sin(3\pi t).}$$

b) Since:

$$\frac{\omega_x}{1} \;=\; \frac{\omega_y}{2} \;=\; \frac{\omega_z}{3} \;=\; \pi,$$

the frequencies are *commensurate* and m traces out a *Lissajous figure* (a closed path), returning periodically to its initial conditions. Since the integers in the denominators have no common factors, the time required for m to complete a single loop and return to its initial conditions, τ, is given by equation 4.4.20 in the text:

$$\boxed{\tau \;=\; 2\pi\frac{1}{\omega_x} \;=\; 2\pi\frac{2}{\omega_y} \;=\; 2\pi\frac{3}{\omega_z} \;=\; 2, \;\; \text{in units where } k=\pi^2 m.}$$

Problem 35.

Solution: Let the position and velocity of the electron be $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$. For an electron, $q = -e$, and the Lorentz force is:

$$\begin{aligned} \vec{F} &= -e(\vec{E} + \vec{v} \times \vec{B}) \;=\; -e\big(E\hat{\jmath} + \dot{x}B\,\underbrace{\hat{\imath} \times \hat{k}}_{-\hat{\jmath}} + \dot{y}B\,\underbrace{\hat{\jmath} \times \hat{k}}_{\hat{\imath}}\big) \;=\; -e\dot{y}B\hat{\imath} + e(\dot{x}B - E)\hat{\jmath} \\ &= m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}) \end{aligned}$$

$$\Rightarrow \quad \ddot{x} \;=\; -\frac{eB}{m}\dot{y} \;\equiv\; -\omega\dot{y}; \qquad \ddot{y} \;=\; \omega\dot{x} - \frac{eE}{m}; \qquad \ddot{z} \;=\; 0, \tag{1}$$

where $\omega = \dfrac{eB}{m}$ is the cyclotron frequency. Integrate over time the first of equations (1) to get:

$$\dot{x} \;=\; -\omega y + c,$$

where c is the constant of integration. Now, at $t = 0$, $y = 0 \;\Rightarrow\; \dot{x}_0 = v_0 = c$. Thus,

$$\dot{x} \;=\; -\omega y + v_0. \tag{2}$$

Substitute (2) into the second of equations (1) to get:

$$\ddot{y} \;=\; -\omega^2 y + \omega v_0 - \frac{eE}{m} \quad \Rightarrow \quad \ddot{y} + \omega^2 y \;=\; \omega v_0 - \frac{eE}{m}. \tag{3}$$

To solve (3), start with the *homogeneous* equation: $\ddot{y}_{\mathrm{H}} + \omega^2 y_{\mathrm{H}} = 0$, whose solution is:

$$y_{\mathrm{H}}(t) \;=\; -a\cos(\omega t + \delta),$$

where a and δ are constants of integration, and where we have inserted the minus sign in anticipation of the final result.

Next, a *particular* solution [*anything* that solves (3)] is evidently:

$$y_{\mathrm{P}} \;=\; \frac{v_0}{\omega} - \frac{eE}{m\omega^2} \;=\; \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right).$$

Thus, the general solution to (3) is:

$$y(t) \;=\; y_{\mathrm{H}}(t) + y_{\mathrm{P}} \;=\; -a\cos(\omega t + \delta) + \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right). \tag{4}$$

Now apply boundary conditions to (4). Thus,

$$\begin{aligned} \dot{y}(0) &= 0 \quad \Rightarrow \quad a\sin\delta \;=\; 0 \quad \Rightarrow \quad \delta \;=\; 0; \\ y(0) &= 0 \quad \Rightarrow \quad a\cos\delta \;=\; a \;=\; \frac{1}{\omega}\left(v_0 - \frac{E}{B}\right), \end{aligned} \tag{5}$$

and (4) becomes:

$$y(t) \;=\; -a\cos\omega t + a \;=\; a(1-\cos\omega t), \tag{6}$$

where a is given by (5). Next, substitute (6) into (2) to get:

$$\dot{x} \;=\; -\omega a(1-\cos\omega t) + v_0,$$

which integrates to:

$$x(t) \;=\; -\omega a\left(t - \frac{1}{\omega}\sin\omega t\right) + v_0 t + \cancelto{0}{x_0} = \; a\sin\omega t + (v_0 - \omega a)t \;=\; a\sin\omega t + bt,$$

where $b = v_0 - \omega a = \dfrac{E}{B}$.

Finally, the last of equations (1) integrates trivially to $z(t) = z_0 + \dot{z}_0 t = 0$ since, from the given initial conditions, $z_0 = 0$ and $\dot{z}_0 = 0$.

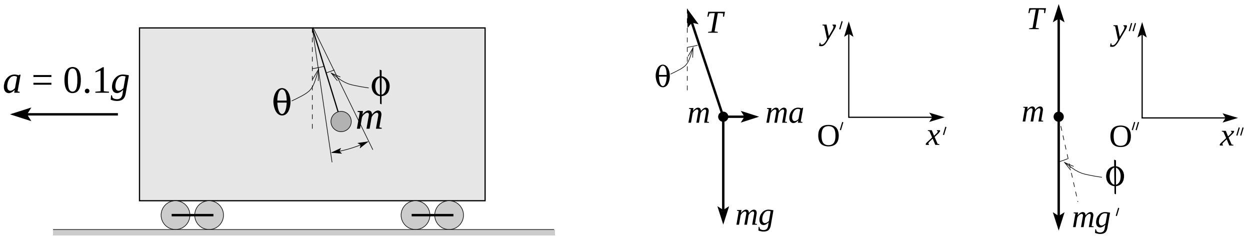
Thus, the path of the electron, as parameterised by t, is given by:

$$\boxed{x(t) \;=\; a\sin\omega t + bt; \qquad y(t) \;=\; a(1-\cos\omega t); \qquad z(t) \;=\; 0,}$$

where $\omega = eB/m$, $b = E/B$, and $a = (v_0 - b)/\omega$, as desired.

Problem 36.

Solution: First, note from the figure that the small angle of oscillation, ϕ, is distinct from the angle θ at which the plumb bob in equilibrium hangs, and which does not need to be small.



In the O′ frame, the net apparent "gravitational" force on m is given by:

$$\vec{g}' \;=\; a\hat{\imath} - g\hat{\jmath} \quad\Rightarrow\quad g' \;=\; g\sqrt{1.01}.$$

If we rotate the coordinate system, O′, by the angle θ to O″, as shown in the second FBD, the situation is identical to that in a non-accelerating frame other than the fact that g'

has been substituted for g. Thus, we can immediately write for the angular frequency of small-amplitude (ϕ) oscillations of the bob:

$$\omega' = \sqrt{\frac{g'}{l}}.$$

Therefore, the period of oscillation of the plumb bob in the accelerating boxcar, τ', is:

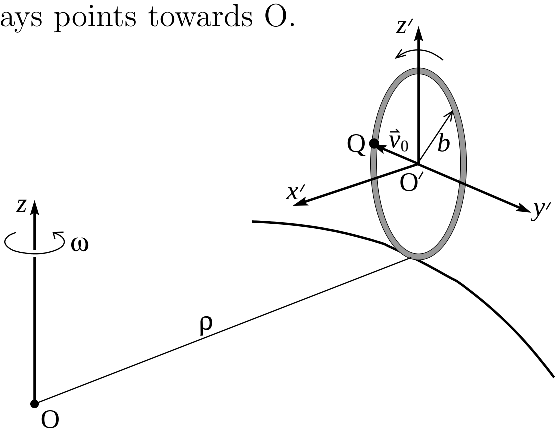
$$\tau' = \frac{2\pi}{\omega'} = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g\sqrt{1.01}}} \simeq 0.9975\, 2\pi\sqrt{\frac{l}{g}}$$

$$\Rightarrow \quad \boxed{\tau' = 0.9975\,\tau,}$$

where l is the length of the cord, and τ is the period of oscillation when the boxcar is not accelerating.

Problem 37.

Solution: This follows closely example 5.2.2 of the text (eds. 6 and 7). As shown in the figure, let the leading point on the wheel be Q, and let the point at the centre of the track be the origin of an inertial coordinate system, O. Then, as we did in class, consider an accelerating coordinate system, O$'$ at the axle of the wheel that does *not* rotate with the wheel. Thus, $\hat{k}'$ always points vertically and $\hat{\imath}'$ always points towards O.



Start with equation (5.2.14) of the text:

$$\vec{a} = \vec{a}' + \dot{\vec{\omega}}\times\vec{r}' + 2\,\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}') + \vec{A}. \quad (1)$$

Since the wheel does not slip, the translational speed of the wheel relative to O, v_0, is also the tangential speed of any point of the rim relative to O$'$. Thus,

Step 1: "Hunting and gathering".

acceleration of Q relative to O$'$:	$\vec{a}' = \frac{v_0^2}{b}\hat{\jmath}'$;
angular velocity of O$'$ relative to O:	$\vec{\omega} = \frac{v_0}{\rho}\hat{k}'$ (not Q about O$'$!);
angular acceleration of O$'$ relative to O:	$\dot{\vec{\omega}} = 0$;
position of Q relative to O$'$:	$\vec{r}' = -b\hat{\jmath}'$;
velocity of Q relative to O$'$:	$\vec{v}' = -v_0\hat{k}'$;
acceleration of O$'$ relative to O:	$\vec{A} = \frac{v_0^2}{\rho}\,\hat{\imath}'$.

Step 2: Do the cross products.

$$\begin{aligned}
&\text{transverse term:} && \dot{\vec{\omega}} \times \vec{r}\,' \;=\; 0; \\
&\text{Coriolis term:} && 2\vec{\omega} \times \vec{v}\,' \;=\; \frac{2v_0^2}{\rho}\,\hat{k}' \times (-\hat{k}') \;=\; 0; \\
&\text{centrifugal term:} && \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') \;=\; -\frac{v_0^2}{\rho^2} b\,\hat{k}' \times (\hat{k}' \times \hat{\jmath}\,') \;=\; \frac{v_0^2}{\rho^2} b\,\hat{\jmath}\,'.
\end{aligned}$$

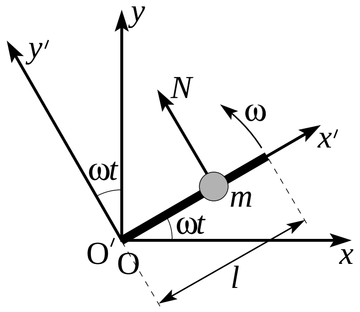
Step 3: Assemble the accelerations: Substituting everything into equation (1), we get:

$$\vec{a} \;=\; \frac{v_0^2}{b}\hat{\jmath}\,' + 0 + 0 + \frac{v_0^2}{\rho^2} b\,\hat{\jmath}\,' + \frac{v_0^2}{\rho}\,\hat{\imath}\,' \quad \Rightarrow \quad \boxed{\vec{a} \;=\; \frac{v_0^2}{\rho}\,\hat{\imath}\,' + v_0^2 b\left(\frac{1}{b^2} + \frac{1}{\rho^2}\right)\hat{\jmath}\,'.}$$

Problem 38.

Solution: a) As in the figure, define a rotating coordinate system, O′, such that $\hat{\imath}\,'$ remains along the rod and such that the origin of O′ coincides with the origin of an inertial coordinate system, O. Then,

Step 1: "Hunting and gathering":



angular velocity of O′ relative to O: $\vec{\omega} = \omega\hat{k}'$

angular acceleration of O′ relative to O: $\dot{\vec{\omega}} = 0;$

position of m relative to O′: $\vec{r}\,' = x'\hat{\imath}\,';$

velocity of m relative to O′: $\vec{v}\,' = \dot{x}'\hat{\imath}\,';$

acceleration of m relative to O′: $\vec{a}\,' = \ddot{x}'\hat{\imath}\,';$

acceleration of O′ relative to O: $\vec{A} = 0.$

Step 2: Do the cross products:

$$\begin{aligned}
&\text{transverse force:} && -m\dot{\vec{\omega}} \times \vec{r}\,' \;=\; 0; \\
&\text{Coriolis force:} && -2m\vec{\omega} \times \vec{v}\,' \;=\; -2m\omega\dot{x}'\hat{k}' \times \hat{\imath}\,' \;=\; -2m\omega\dot{x}'\hat{\jmath}\,'; \\
&\text{centrifugal force:} && -m\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') \;=\; -m\omega^2 x'\,\hat{k}' \times (\hat{k}' \times \hat{\imath}\,') \;=\; m\omega^2 x'\hat{\imath}\,'.
\end{aligned}$$

Step 3: Assemble the forces: From the figure, the only real force acting in the x'-y' plane is the normal force: $\vec{F} = N\hat{\jmath}\,'$. Substituting this and the inertial forces into equation (5.3.2) of the text:

$$\vec{F}\,' \;=\; \vec{F} - m\dot{\vec{\omega}} \times \vec{r}\,' - 2m\vec{\omega} \times \vec{v}\,' - m\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') - \,m\vec{A} \;=\; m\vec{a}\,',$$

we get:

$$m\omega^2 x'\hat{\imath}' + (N - 2m\omega\dot{x}')\hat{\jmath}' \;=\; m\ddot{x}'\hat{\imath}'.$$

Since we are being asked about the motion along the rod, we only need consider the $\hat{\imath}'$ component of this vector equation. Thus,

$$\omega^2 x' \;=\; \ddot{x}' \quad \Rightarrow \quad x'(t) \;=\; Ae^{\omega t} + Be^{-\omega t} \quad \text{and} \quad \dot{x}'(t) \;=\; A\omega e^{\omega t} - B\omega e^{-\omega t}.$$

Imposing boundary conditions,

$$\left.\begin{array}{lll} x'(0) \;=\; \dfrac{l}{2} & \Rightarrow & A + B = \dfrac{l}{2} \\ \dot{x}'(0) \;=\; 0 & \Rightarrow & A - B = 0 \end{array}\right\} \quad \Rightarrow \quad A \;=\; B \;=\; \frac{l}{4},$$

and we have:

$$\boxed{x'(t) \;=\; \frac{l}{4}\left(e^{\omega t} + e^{-\omega t}\right) \;=\; \frac{l}{2}\cosh\omega t.} \tag{1}$$

b) To determine when the bead reaches the end of the rod,

$$x'(t) \;=\; l \;=\; \frac{l}{4}\left(e^{\omega t} + e^{-\omega t}\right) \quad \Rightarrow \quad e^{\omega t} + e^{-\omega t} - 4 \;=\; 0$$

$$\Rightarrow \quad e^{2\omega t} - 4e^{\omega t} + 1 \;=\; 0 \quad \Rightarrow \quad e^{\omega t} \;=\; \frac{4 \pm \sqrt{16 - 4}}{2} \;=\; 2 \pm \sqrt{3}.$$

For $\omega t > 0$, $e^{\omega t} > 1$ and we choose the positive root to get:

$$\omega t \;=\; \ln(2 + \sqrt{3}) \quad \Rightarrow \quad \boxed{t \;\simeq\; \frac{1.317}{\omega}.}$$

c) Finally, differentiating equation (1), we get:

$$v'(t) \;=\; \frac{l\omega}{4}\left(e^{\omega t} - e^{-\omega t}\right).$$

When the bead leaves the rod, $e^{\omega t} = 2 + \sqrt{3}$, and thus:

$$v'\big|_{x'=l} \;=\; \frac{l\omega}{4}\left(2 + \sqrt{3} - \frac{1}{2 + \sqrt{3}}\,\frac{2 - \sqrt{3}}{2 - \sqrt{3}}\right) \;=\; \frac{l\omega}{4}\left(2 + \sqrt{3} - 2 + \sqrt{3}\right)$$

$$\Rightarrow \quad \boxed{v'(l) \;=\; l\omega\frac{\sqrt{3}}{2}.}$$

Problem 39.

Solution: a) In a coordinate system O$'$ fixed to the ground in which x' points east, y' points north, and z' points upward along the plumb line, $\vec{v}' = v_0\hat{\jmath}'$ and $\vec{\omega} = \omega(\cos\lambda\hat{\jmath}' + \sin\lambda\hat{k}')$ woth $v_0 = 180\,\mathrm{m\,s^{-1}}$ and $\omega = 7.292 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$. Thus,

$$\begin{aligned} \vec{F}_{\text{Cor}} &= -2m\vec{\omega} \times \vec{v}' = -2m\omega v_0(\cos\lambda\,\hat{\jmath}' \times \hat{\jmath}' + \sin\lambda\,\hat{k}' \times \hat{\jmath}') \\ &= 2m\omega v_0 \sin\lambda\,\hat{\imath}' = \underline{\underline{m(0.0172\,\mathrm{m\,s^{-2}})\text{ due east.}}} \end{aligned}$$

b) As a ratio of its weight, the Coriolis force is:

$$\frac{F_{\text{Cor}}}{mg} = \frac{2\cancel{m}\omega v_0 \sin\lambda}{\cancel{m}g} = \underline{\underline{1.76 \times 10^{-3},}}$$

using $g = 9.81\,\mathrm{m\,s^{-1}}$. For a 2,000 pound car, this represents a force of about 3.5 pounds.

Problem 40.

Solution: From class, we found the east-west and vertical coordinates, (x', z'), of a projectile near the surface of the earth are:

$$\left.\begin{aligned} x'(t) &= \dot{x}'_0 t - \omega t^2(\dot{z}'_0\cos\lambda - \dot{y}'_0\sin\lambda) + \frac{\omega g t^3}{3}\cos\lambda; \\ z'(t) &= \dot{z}'_0 t - \tfrac{1}{2}gt^2 + \omega t^2 \dot{x}'_0 \cos\lambda, \end{aligned}\right\} \qquad (2)$$

taking the initial position $(x'_0, z'_0) = (0, 0)$. Since the initial velocity is directed eastward at an elevation angle α,

$$\vec{v}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0) = (v_0\cos\alpha, 0, v_0\sin\alpha),$$

and equations (2) become:

$$x'(t) = v'_0 t\cos\alpha + \omega t^2\cos\lambda\left(\frac{gt}{3} - v'_0\sin\alpha\right); \qquad (3)$$

$$z'(t) = v'_0 t\sin\alpha - \tfrac{1}{2}gt^2 + \omega t^2 v'_0\cos\alpha\cos\lambda. \qquad (4)$$

a) When the bullet hits the ground, $z'(t) = 0$ and equation (4) requires,

$$t\big(v'_0\sin\alpha - \tfrac{1}{2}gt + \omega t v'_0\cos\alpha\cos\lambda\big) = t\big[v'_0\sin\alpha - t\big(\tfrac{1}{2}g - \omega v'_0\cos\alpha\cos\lambda\big)\big] = 0$$

$$\Rightarrow \quad t = 0 \quad \text{or} \quad \frac{2v'_0\sin\alpha}{g - 2\omega v'_0\cos\alpha\cos\lambda},$$

corresponding to the beginning and the end of the trajectory respectively. Thus,

$$\boxed{t = \frac{2v'_0\sin\alpha}{g}\left(1 - \frac{2\omega v'_0}{g}\cos\alpha\cos\lambda\right)^{-1},}$$

which is equation (1).

b) Assuming $\omega v_0' \ll g$, the second term in the binomial is $\ll 1$, and we may use a binomial expansion to get:

$$\left(1-\frac{2\omega v_0'}{g}\cos\alpha\cos\lambda\right)^{-1} = 1+\frac{2\omega v_0'}{g}\cos\alpha\cos\lambda+\cdots\,,$$

retaining just the first two terms. Thus, equation (1) becomes:

$$t \approx \frac{2v_0'\sin\alpha}{g}\left(1+\frac{2\omega v_0'}{g}\cos\alpha\cos\lambda\right) = \frac{2v_0'\sin\alpha}{g}+\frac{2v_0'\sin\alpha}{g}\,\frac{2\omega v_0'}{g}\cos\alpha\cos\lambda$$

$$= \frac{2v_0'\sin\alpha}{g}+\underbrace{\frac{{v_0'}^2\,\overbrace{2\sin\alpha\cos\alpha}^{\sin 2\alpha}}{g}}_{R_0}\,\frac{2\omega\cos\lambda}{g}$$

$$\Rightarrow \quad \boxed{t \approx \frac{2v_0'\sin\alpha}{g}+\frac{2\omega R_0\cos\lambda}{g},} \tag{5}$$

as desired.

c) Finally, for the range, R_0', we use equation (3) evaluated at time t. Now, the first term in equation (3) has no factor ω, and so we must include both terms in equation (5) when substituting for t. However, the second term in equation (3) is already proportional to ω, and thus when substituting for t, we may retain just the leading term in equation (5). Bearing all this in mind, we have:

$$x'(t) \approx v_0'\left(\frac{2v_0'\sin\alpha}{g}+\frac{2\omega R_0\cos\lambda}{g}\right)\cos\alpha+\omega\left(\frac{2v_0'\sin\alpha}{g}\right)^2\cos\lambda\left(\underbrace{\frac{\cancel{g}}{3}\,\frac{2v_0'\sin\alpha}{\cancel{g}}-v_0'\sin\alpha}_{-\frac{1}{3}v_0'\sin\alpha}\right)$$

$$= \underbrace{\frac{{v_0'}^2\sin 2\alpha}{g}}_{R_0}+\omega\cos\lambda\left(R_0\frac{2v_0'\cos\alpha}{g}-\frac{4{v_0'}^3\sin^3\alpha}{3g^2}\right)$$

$$= R_0+\omega\cos\lambda\left(R_0\sqrt{\frac{2\cos\alpha}{g\sin\alpha}}\,\underbrace{\frac{v_0'\sqrt{2\sin\alpha\cos\alpha}}{\sqrt{g}}}_{\sqrt{R_0}}-\frac{1}{3}\sqrt{\frac{2}{g}}\,\underbrace{\frac{{v_0'}^3(2\sin\alpha\cos\alpha)^{3/2}}{g^{3/2}}}_{R_0^{3/2}}\,\frac{\sin^{3/2}\alpha}{\cos^{3/2}\alpha}\right)$$

$$\Rightarrow \quad \boxed{x'(t) = R_0' = R_0+\sqrt{\frac{2R_0^3}{g}}\,\omega\cos\lambda\left(\cot^{1/2}\alpha-\frac{1}{3}\tan^{3/2}\alpha\right),}$$

as desired.

If you put in the numbers, for a rifle with a muzzle speed of $500\,\mathrm{m\,s^{-1}}$ aimed at an elevation angle of $\alpha = 30°$, the range without taking into account the earth's rotation is $R_0 = 22{,}070\,\mathrm{m}$.

The correction term proportional to $\omega = 7.292 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$ amounts to an additional $89.3\,\mathrm{m}$ at latitude $\lambda = 45^\circ\,\mathrm{N}$, or about 0.4%. In fact, atmospheric effects are a much more important modifier to R_0 than the effects of the earth's rotation.
