Solutions to Tutorial 1

PHYS 2302 (Mechanics I); D. A. Clarke

Tutorial 1.1

Problem 1 In some Cartesian coordinate system, let $\vec{A} = (3, -12, 4)$ and $\vec{B} = 4\hat{\imath} + 4\hat{\jmath} - 7\hat{k}$. Evaluate:

a) $\vec{A} + \vec{B}$ in unit vector notation;

b) $\vec{A} - \vec{B}$ as an ordered triple;

c) $\vec{A} \times \vec{B}$ in unit vector notation;

d) $\vec{A} \cdot \vec{B}$;

e) the angle between $\vec{A}$ and $\vec{B}$; and

f) the projection of $\vec{A}$ onto $\vec{B}$.

Solution: a) In unit vector notation, $\vec{A} = 3\hat{\imath} - 12\hat{\jmath} + 4\hat{k}$, and thus,

$$\begin{aligned}\vec{A} + \vec{B} &= \big(3\hat{\imath} - 12\hat{\jmath} + 4\hat{k}\big) + \big(4\hat{\imath} + 4\hat{\jmath} - 7\hat{k}\big) = (3+4)\hat{\imath} + (-12+4)\hat{\jmath} + (4-7)\hat{k} \\ &= \underline{\underline{7\hat{\imath} - 8\hat{\jmath} - 3\hat{k}}}.\end{aligned}$$

b) As an ordered triple, $\vec{B} = (4, 4, -7)$, and thus,

$$\vec{A} - \vec{B} = (3, -12, 4) - (4, 4, -7) = (3-4, -12-4, 4+7) = \underline{\underline{(-1, -16, 11)}}.$$

c) Starting with the ordered triples,

$$\begin{aligned}\vec{A} \times \vec{B} &= (3, -12, 4) \times (4, 4, -7) \\ &= \big((-12)(-7) - (4)(4), (4)(4) - (3)(-7), (3)(4) - (-12)(4)\big) \\ &= (84 - 16, 16 + 21, 12 + 48) = (68, 37, 60) = \underline{\underline{68\hat{\imath} + 37\hat{\jmath} + 60\hat{k}}}.\end{aligned}$$

Alternately, starting with unit vector notation,

$$\begin{aligned}\vec{A} \times \vec{B} &= \big(3\hat{\imath} - 12\hat{\jmath} + 4\hat{k}\big) \times \big(4\hat{\imath} + 4\hat{\jmath} - 7\hat{k}\big) \\ &= 12\cancelto{0}{(\hat{\imath}\times\hat{\imath})} + 12\cancelto{\hat{k}}{(\hat{\imath}\times\hat{\jmath})} - 21\cancelto{-\hat{\jmath}}{(\hat{\imath}\times\hat{k})} - 48\cancelto{-\hat{k}}{(\hat{\jmath}\times\hat{\imath})} - 48\cancelto{0}{(\hat{\jmath}\times\hat{\jmath})} + 84\cancelto{\hat{\imath}}{(\hat{\jmath}\times\hat{k})} \\ &\qquad + 16\cancelto{\hat{\jmath}}{(\hat{k}\times\hat{\imath})} + 16\cancelto{-\hat{\imath}}{(\hat{k}\times\hat{\jmath})} - 28\cancelto{0}{(\hat{k}\times\hat{k})}\end{aligned}$$

$$= (84-16)\hat{\imath} + (21+16)\hat{\jmath} + (12+48)\hat{k} \;=\; \underline{\underline{68\hat{\imath} + 37\hat{\jmath} + 60\hat{k}}}.$$

d) Starting with the ordered triples,

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (3,-12,4)\cdot(4,4,-7) \;=\; (3)(4)+(-12)(4)+(4)(-7) \;=\; 12-48-28\\ &= \underline{\underline{-64}}.\end{aligned}$$

e) We first need the magnitudes of $\vec{A}$ and $\vec{B}$:

$$\begin{aligned}A &= \sqrt{\vec{A}\cdot\vec{A}} \;=\; \sqrt{9+144+16} \;=\; \sqrt{169} \;=\; 13; \quad \text{and}\\ B &= \sqrt{\vec{B}\cdot\vec{B}} \;=\; \sqrt{16+16+49} \;=\; \sqrt{81} \;=\; 9.\end{aligned}$$

We can then write,

$$\cos\theta \;=\; \frac{\vec{A}\cdot\vec{B}}{AB} \;=\; \frac{-57}{(13)(9)} \;=\; -\frac{19}{39} \sim -0.4872 \quad\Rightarrow\quad \underline{\underline{\theta \sim 119.2^\circ}}.$$

f) The projection of $\vec{A}$ onto $\vec{B}$ is given by,

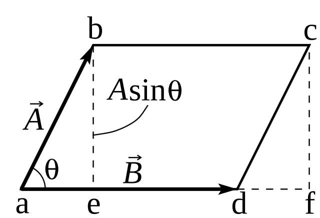
$$A\cos\theta \;=\; (\cancel{13})\frac{-57}{(\cancel{13})(9)} \;=\; \underline{\underline{-\frac{19}{3}}}.$$

If $\vec{A}$ and $\vec{B}$ are attached at their ends and you "drop a perpendicular" from the tip of $\vec{A}$ to $\vec{B}$ (a line that intersects $\vec{B}$ at right angles), then 19/3 is the distance between where the perpendicular intersects $\vec{B}$ and the end of $\vec{B}$. The negative sign means the projection considered as a vector points in the opposite direction of $\vec{B}$ (angle between $\vec{A}$ and $\vec{B}$ is more than 90°).

Problem 2 (FC 1.10) Show that the area of the parallelogram defined by vectors $\vec{A}$ and $\vec{B}$ is $|\vec{A}\times\vec{B}|$.

Solution: Because of the equivalence of triangles abe and dcf,

$$\begin{aligned}\text{area of parallelogram abcd} &= \text{area of rectangle ebcf}\\ &= A\sin\theta B\\ &= |\vec{A}\times\vec{B}|, \text{ as desired.}\end{aligned}$$



Tutorial 1.2

Problem 3 (FC 1.17 and 1.20)

a) A moth follows the elliptical path: $\vec{r}(t) = b\cos\omega t\,\hat{\imath} + 2b\sin\omega t\,\hat{\jmath}$ around a light bulb, where b and ω are constant. Using Cartesian coordinates, find the moth's speed, $v(t)$, and then from this find v at $t = 0$ and $t = \pi/(2\omega)$ (when the moth is at the minimum and maximum distances from the light bulb).

b) A fly follows a helical path, given in Cartesian coordinates by:

$$\vec{r}(t) = b\sin\omega t\,\hat{\imath} + b\cos\omega t\,\hat{\jmath} + ct^2\hat{k},$$

where b, c, and ω are constant. Find $r(t)$, $v(t)$, and $a(t)$ in cylindrical coordinates, and then show that the magnitude of the fly's acceleration, a, is constant.

Solution: a) First,

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -b\omega\sin\omega t\,\hat{\imath} + 2b\omega\cos\omega t\,\hat{\jmath}.$$

Then,

$$v(t) = \sqrt{\vec{v}\cdot\vec{v}} = b\omega\sqrt{\sin^2\omega t + 4\cos^2\omega t} = \boxed{b\omega\sqrt{1 + 3\cos^2\omega t}.}$$

At $t = 0$, $v = 2b\omega$; at $t = \pi/2\omega$, $v = b\omega$.

b) To convert the fly's path to cylindrical coordinates,

$$R = \sqrt{x^2 + y^2} = \sqrt{(b\sin\omega t)^2 + (b\cos\omega t)^2} = b\underbrace{\sqrt{\sin^2\omega t + \cos^2\omega t}}_{1} = b.$$

$$\Rightarrow \quad r(t) = b\hat{e}_R + ct^2\hat{e}_z.$$

Here, the fly is at $(0, b, 0)$—the $+y$-axis— at $t = 0$ and flies in a clockwise sense with $\phi = \omega t$. Then, from equations 1.12.2 and 1.12.3 in the text:

$$\vec{v}(t) = b\omega\hat{e}_\phi + 2ct\hat{e}_z$$

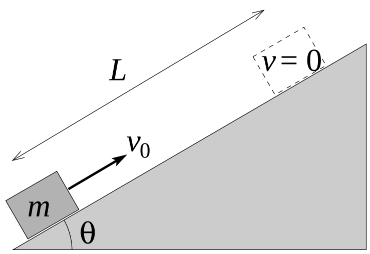
$$\vec{a}(t) = -b\omega^2\hat{e}_R + 2c\hat{e}_z.$$

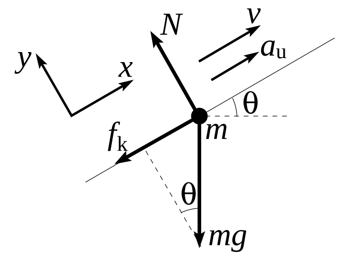
Thus, $\vec{a}\cdot\vec{a} = b^2\omega^4 + 4c^2 =$ constant, as desired.

Tutorial 1.3

Problem 4 (FC 2.10) A block of mass m is shoved up a sloped plane with initial velocity v_0. If the inclination angle of the plane is $\theta = 30°$ and the coefficient of kinetic friction between the block and plane is $\mu_k = 0.1$, find in terms of v_0 how long it takes for the block to slide back down to where it started. You may assume that the static coefficient of friction is insufficient to stop the block at the top of its rise.

Solution:





On the way up, we have from the FBD:

$$x/\quad -f_\mathrm{k} - mg\sin\theta \;=\; ma_\mathrm{u}; \tag{1}$$

$$y/\quad N - mg\cos\theta \;=\; 0 \quad\Rightarrow\quad f_\mathrm{k} \;=\; \mu_\mathrm{k}N \;=\; \mu_\mathrm{k}mg\cos\theta. \tag{2}$$

Substitute equation (2) into equation (1) to get:

$$a_\mathrm{u} \;=\; -g(\mu_\mathrm{k}\cos\theta + \sin\theta).$$

Now, at a constant acceleration, a_u, we can use the equations of kinematics. Thus, the time for m to reach the top of its trajectory is:

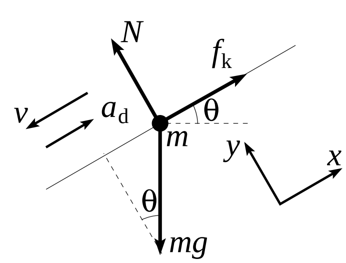
$$t_\mathrm{u} \;=\; \frac{\cancelto{0}{v} - v_0}{a_\mathrm{u}} \;=\; \frac{v_0}{g(\mu_\mathrm{k}\cos\theta + \sin\theta)},$$

and the distance travelled up the plane, L_u, is given by:

$$L_\mathrm{u} \;=\; x - x_0 \;=\; \frac{\cancelto{0}{v^2} - v_0^2}{2a_\mathrm{u}} \;=\; \frac{v_0^2}{2g(\mu_\mathrm{k}\cos\theta + \sin\theta)}. \tag{3}$$

Next, on the way down, we have from the FBD in the x-direction:

$$f_\mathrm{k} - mg\sin\theta \;=\; ma_\mathrm{d}. \tag{4}$$



Substitute equation (2) into equation (4) to get:

$$a_\mathrm{d} \;=\; -g(\sin\theta - \mu_\mathrm{k}\cos\theta). \tag{5}$$

At constant acceleration, a_d, the distance back down the plane is $L_\mathrm{d} = -L_\mathrm{u}$, given by:

$$L_\mathrm{d} \;=\; \cancelto{0}{v_i t} + \tfrac{1}{2}a_\mathrm{d}t_\mathrm{d}^2 \;=\; -L_\mathrm{u} \;=\; -\frac{v_0^2}{2g(\mu_\mathrm{k}\cos\theta + \sin\theta)},$$

from equation (3). Thus, with the help of equation (5),

$$t_\mathrm{d}^2 \;=\; -\frac{1}{a_\mathrm{d}}\frac{v_0^2}{g(\mu_\mathrm{k}\cos\theta + \sin\theta)} \;=\; \frac{v_0^2}{g^2(\sin^2\theta - \mu_\mathrm{k}^2\cos^2\theta)} \quad\Rightarrow\quad t_\mathrm{d} \;=\; \frac{v_0}{g\sqrt{\sin^2\theta - \mu_\mathrm{k}^2\cos^2\theta}}.$$

Therefore, the total time for the block to return to its initial position is:

$$
\begin{aligned}
t = t_\mathrm{u} + t_\mathrm{d} &= \frac{v_0}{g(\mu_\mathrm{k}\cos\theta + \sin\theta)} + \frac{v_0}{g\sqrt{\sin^2\theta - \mu_\mathrm{k}^2\cos^2\theta}} \\
&= 0.1738\,v_0 + \underbrace{0.2070}_{\text{down trip takes longer than up trip}}v_0 \\
&= 0.381\,v_0,
\end{aligned}
$$

using $g = 9.81\ \mathrm{m\,s^{-2}}$, $\theta = 30°$, and $\mu_\mathrm{k} = 0.1$. Note that implicit in this calculation is the assumption that the static friction is insufficient to hold the block at the top of its trajectory where it momentarily comes to rest. Thus, we assume that $\mu_\mathrm{s} < \tan 30° = 0.577$. For $\mu_\mathrm{k} = 0.1$, it is *probably* a safe bet that $\mu_\mathrm{s} < 0.577$, but not necessarily so.

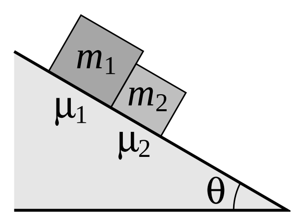
Tutorial 1.4

Problem 4 Two blocks of masses m_1 and m_2 remain in contact as they slide down a ramp inclined at an angle θ to the horizontal. The coefficient of kinetic friction between each of the blocks and the ramp is μ_1 and μ_2 respectively.

a) What is the normal force acting between the two masses and what is the criterion that it be greater than zero?

b) What is their common acceleration down the ramp?

c) If $\mu_2 = 2\mu_1$ and $m_2 = \frac{1}{2}m_1$, what is the maximum value for μ_1 if $\theta = 30°$?



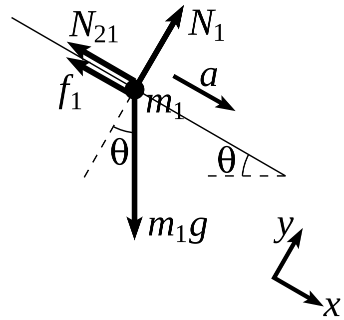
Solution: From the free body diagrams, we have from Newton's second law:

$$x/ \quad -f_1 - N_{21} + m_1 g \sin\theta = m_1 a; \tag{1}$$

$$y/ \quad N_1 - m_1 g \cos\theta = 0; \tag{2}$$

$$x/ \quad -f_2 + N_{12} + m_2 g \sin\theta = m_2 a; \tag{3}$$

$$y/ \quad N_2 - m_2 g \cos\theta = 0, \tag{4}$$

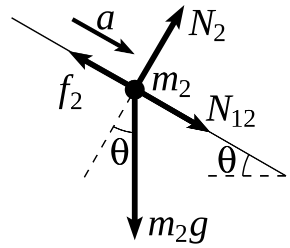


where the kinetic friction forces are given by:

$$f_1 = \mu_1 N_1 = \mu_1 m_1 g \cos\theta; \quad f_2 = \mu_2 N_2 = \mu_2 m_2 g \cos\theta, \tag{5}$$

using equations (2) and (4). Further, from Newton's third law, N_{21} and N_{12} are an action-reaction pair, and thus,

$$N_{21} = N_{12} = N. \tag{6}$$



Substituting Eq. (5) and (6) into Eq. (1) and (3), we get,

$$-\mu_1 m_1 g \cos\theta - N + m_1 g \sin\theta = m_1 a; \tag{7}$$

$$-\mu_2 m_2 g \cos\theta + N + m_2 g \sin\theta = m_2 a. \tag{8}$$

a) To find the normal force, N, divide Eq. (8) by m_2, Eq. (7) by m_1, and subtract:

$$-\mu_1 g \cos\theta - \frac{N}{m_1} + \cancel{g\sin\theta} - \left(-\mu_2 g \cos\theta + \frac{N}{m_2} + \cancel{g\sin\theta}\right) = \cancel{a} - \cancel{a}$$

$$\Rightarrow \quad g\cos\theta(\mu_2 - \mu_1) - N\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = 0 \quad \Rightarrow \quad \boxed{N = \mu g \cos\theta(\mu_2 - \mu_1),} \tag{9}$$

where

$$\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} = \frac{m_1 m_2}{m_1 + m_2},$$

is the *reduced mass* of the two-mass system. We'll see the reduced mass crop up numerous times in PHYS 2303. Thus, for $N > 0$ in Eq. 9, clearly we must have,

$$\boxed{\mu_2 > \mu_1,}$$

which makes sense, since under gravity all masses fall at the same acceleration, even on a slope. (Galileo himself proved that!) The only way for $N > 0$ is if m_1 wants to fall faster than m_2, and this will be true only if the frictional force felt by m_2 is greater than that felt by m_1.

b) To find the acceleration, a, add Eq. (7) and (8) to get,

$$-g\cos\theta(\mu_1 m_1 + \mu_2 m_2) + g\sin\theta(m_1 + m_2) = a(m_1 + m_2)$$

$$\Rightarrow \quad a = -g\cos\theta\,\frac{\mu_1 m_1 + \mu_2 m_2}{m_1 + m_2} + g\sin\theta \quad \Rightarrow \quad \boxed{a = g\cos\theta\left(\tan\theta - \frac{\mu_1 m_1 + \mu_2 m_2}{m_1 + m_2}\right).}$$

Evidently, for $a > 0$, we must have,

$$\boxed{\tan\theta > \frac{\mu_1 m_1 + \mu_2 m_2}{m_1 + m_2}.} \tag{10}$$

c) If $\mu_2 = 2\mu_1$ and $m_2 = \frac{1}{2}m_1$, Eq. (10) becomes,

$$\tan\theta > \frac{\mu_1 \cancel{m_1} + 2\mu_1 \frac{1}{2}\cancel{m_1}}{\cancel{m_1} + \frac{1}{2}\cancel{m_1}} = \frac{2\mu_1}{3/2} = \frac{4}{3}\mu_1.$$

Thus, for $\theta = 30^\circ$,

$$\boxed{\mu_1 < \frac{3}{4}\tan\theta \sim 0.4330.}$$
