

Solutions to Tutorial 2

PHYS 2302 (Mechanics I); D. A. Clarke

Tutorial 2.1

Problem 1 Solve each of the first order ODEs for $y(x)$ using separation of variables:

a) $4x \frac{dy}{dx} - y(x-3) = 0;$

b) $\cos x \frac{dy}{dx} + y \sin x = 0;$

c) $(x^2 - 4) \frac{dy}{dx} + y = -1.$

Solution: a) Separating the variables, we get:

$$\frac{dy}{dx} = \frac{y(x-3)}{4x} \Rightarrow 4 \frac{dy}{y} = \frac{x-3}{x} dx,$$

where we can now integrate:

$$4 \int \frac{dy}{y} = 4 \ln y = \int \left(1 - \frac{3}{x}\right) dx = x - 3 \ln x + c.$$

Here, c is the combined constant of integration from both integrals. Solving for y , we get:

$$e^{4 \ln y} = e^{\ln y^4} = y^4 = e^{x-3 \ln x+c} = e^c x^{-3} e^x \Rightarrow \boxed{y(x) = C x^{-3/4} e^{x/4}},$$

where $C = e^{c/4}$ is a “constant of integration”.

b) Separating the variables, we get:

$$\frac{dy}{dx} = -\frac{y \sin x}{\cos x} \Rightarrow \frac{dy}{y} = -\frac{\sin x dx}{\cos x}.$$

Letting $z = \cos x$ (and thus $dz = -\sin x dx$), we get,

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \ln y = \ln z + c \Rightarrow y = e^{\ln z+c} = e^c z \Rightarrow \boxed{y(x) = C \cos x},$$

where $C = e^c$ is a “constant of integration”.

c) Separating the variables, we get:

$$\frac{dy}{dx} = -\frac{y+1}{x^2-4} \Rightarrow \frac{dy}{y+1} = -\frac{dx}{x^2-4} = \frac{1}{4} \frac{dx}{x+2} - \frac{1}{4} \frac{dx}{x-2},$$

using partial fractions. This can now be integrated directly to get:

$$\ln(y+1) = \frac{1}{4}(\ln(x+2) - \ln(x-2)) = \frac{1}{4} \ln\left(\frac{x+2}{x-2}\right) + c = \ln\left(\frac{x+2}{x-2}\right)^{1/4} + c.$$

Solving for y , we get:

$$y+1 = e^c \left(\frac{x+2}{x-2}\right)^{1/4} \Rightarrow \boxed{y(x) = C \left(\frac{x+2}{x-2}\right)^{1/4} - 1,}$$

where $C = e^c$ is a “constant of integration”.

Notice how often exponentials are involved in solutions to ODEs? Good thing to keep in mind...

Tutorial 2.2

Problem 2 A baseball ($m = 0.145$ kg) is dropped from rest from the top of the CN tower ($h = 553$ m) and hits the ground at its *terminal velocity*, $v_t = 42.5$ m s⁻¹ (where forces of gravity and air drag balance).

- How much work is done by gravity?
- How much work is done by air drag?
- If there were no air drag, what would be the speed of the ball when it hits the ground?

Note: For rectilinear motion, work done by a force, F , acting in the same (+) or opposite (-) direction as the displacement between x_0 and x is:

$$W_F = \pm \int_{x_0}^x F dx'. \quad (1)$$

Solution: For this problem, we can evaluate the work done by a force directly when it depends only upon position (such as gravity), and use the work-kinetic theorem to find the work done by a force not exclusively dependent upon position such as air drag.

a) The gravitational force, mg , acts in the same direction as the displacement, and equation (1) becomes:

$$W_{mg} = \int_0^h mg dz = mgz \Big|_0^h = mgh \sim \underline{\underline{786.6 \text{ J}}}.$$

The fact $W_{mg} > 0$ indicates gravity works to accelerate m .

b) The drag force, D , acts in the opposite direction as the displacement, and equation (1) becomes:

$$W_D = - \int_0^h D dz.$$

Since D does not depend upon z alone, we can't evaluate the integral directly. Instead, we use the work-kinetic theorem to evaluate W_D . Thus,

$$\begin{aligned} \sum W &= W_{mg} + W_D = \Delta K = K_f - K_i^0 = \frac{1}{2}mv_f^2 \sim 131.0 \text{ J} \\ \Rightarrow W_D &= \Delta K - W_{mg} \sim 131.0 - 786.6 = \underline{\underline{-655.7 \text{ J}}}. \end{aligned}$$

The fact that $W_D < 0$ indicates air drag works to decelerate m .

c) In the absence of air drag, the only force acting on m is gravity which depends only on position. Thus, we can use conservation of energy to get:

$$E_f = K_f + U_f^0 = E_i = K_i^0 + U_i,$$

since m starts from rest ($K_i = 0$) and we define the potential energy on the ground to be 0 ($U_f = 0$). Thus,

$$\frac{1}{2}mv_f^2 = mgh \Rightarrow v_f = \sqrt{2gh} \sim \underline{\underline{104.2 \text{ m s}^{-1}}},$$

about 30% the speed of sound and about 2.5 times the terminal speed (speed with air drag).

Problem 3 (FC 2.17)

a) If $F(x, v) = f(x)g(v)$, show that the differential equation of motion,

$$F(x, v) = m\ddot{x}, \tag{1}$$

can be solved for $x(t)$ by separation of variables and thus direct integration.

b) Can separation of variables be used to solve Eq. (1) if $F(v, t) = g(v)h(t)$? What if $F(x, t) = f(x)h(t)$?

Solution: a) Starting with the equation of motion, we have:

$$F(x, v) = f(x)g(v) = m\ddot{x} = m \frac{d\dot{x}}{dt} = m \frac{dx}{dt} \frac{d\dot{x}}{dx} = mv \frac{dv}{dx},$$

which can be separated by multiplying through by $dx/g(v)$:

$$f(x)dx = \frac{mv}{g(v)}dv \quad \Rightarrow \quad \int f(x)dx = m \int \frac{v}{g(v)}dv. \quad (2)$$

In principle, both integrals are doable (functions of just one variable) and one can find $v(x)$ by direct integration. From this, we write:

$$v(x) = \frac{dx}{dt} \quad \Rightarrow \quad dt = \frac{dx}{v(x)} \quad \Rightarrow \quad \int dt = \int \frac{dx}{v(x)},$$

doing another round of separation of variables, resulting in two integrals. Again, both integrals are, in principle, doable giving us $x(t)$ by direct integration, the final solution to equation (1).

b) Starting this time with,

$$F(v, t) = g(v)h(t) = m\ddot{x} = m\frac{dv}{dt}, \quad (3)$$

once again we have a separable ODE which may be solved by direct integration:

$$h(t)dt = \frac{mdv}{g(v)} \quad \Rightarrow \quad \int h(t)dt = m \int \frac{dv}{g(v)}.$$

Solving these integrals gives, in principle, $v(t)$ from which we write:

$$v(t) = \frac{dx}{dt} \quad \Rightarrow \quad v(t)dt = dx \quad \Rightarrow \quad \int v(t)dt = \int dx,$$

which gives $x(t)$, the solution to equation (1).

Finally, starting with,

$$F(x, t) = f(x)h(t) = m\ddot{x} = m\frac{dv}{dt} \quad \Rightarrow \quad h(t)dt = \frac{m}{f(x)}dv = \frac{m}{f(x)}\frac{dv}{dx}dx. \quad (4)$$

No matter what we do, the equation of motion cannot be expressed in terms of just two variables (*e.g.*, x, v in equation 2; t, v in equation 3), and thus cannot be cleanly separated nor integrated directly. For example, trying to integrate equation (4) directly, we get:

$$\int h(t)dt = m \int \frac{dv}{dx} \frac{dx}{f(x)}.$$

While the LHS is integrable, the RHS is not, unless we know *a priori* $v(x)$. Thus, methods other than separation of variables are needed to solve the equation of motion in this case.