Solutions to Tutorial 3

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Tutorial 3.1

Problem 1 Consider the following linear, second-order, homogeneous ODE:

$$\frac{d^2y}{dx^2} - 4y = 0. \tag{1}$$

a) Find "by inspection" two linearly independent solutions to equation (1).

b) From your two linearly independent solutions, write down the general solution.

c) Show that when the boundary conditions $y(0) = 0$ and $y'(0) = 1$ are applied to your general solution in part b, you get:

$$y(x) = \tfrac{1}{2}\sinh 2x,$$

where $\sinh z \equiv \frac{1}{2}(e^z - e^{-z})$ is the *hyperbolic sine function*, which we'll meet in an upcoming class.

Solution: a) To solve an ODE by inspection requires that it be particularly simple, and equation (1) is such a case.

We rearrange equation (1) to:

$$\frac{d^2y}{dx^2} = 4y,$$

and ask the question: *What function, $y(x)$, has a second derivative equal to itself times a constant?* (in this case 4).

Dropping the "times a constant" bit for the moment, a function whose second derivative is itself is an exponential:

$$\frac{d^2e^x}{dx^2} = \frac{d}{dx}\frac{de^x}{dx} = \frac{d}{dx}e^x = e^x.$$

Thus, solutions to equation (1) are very likely exponentials.

Bringing back the "times a constant" requirement, we modify our trial solution to $y(x) = e^{ax}$ to get:

$$y''(x) = \frac{d^2e^{ax}}{dx^2} = \frac{d}{dx}\frac{de^{ax}}{dx} = \frac{d}{dx}ae^{ax} = a\frac{d}{dx}e^{ax} = a^2e^{ax} = a^2y(x).$$

To customise this to equation (1), we set $a^2 = 4 \Rightarrow a = \pm 2$, and our two solutions are revealed:

$$\boxed{y_1(x) = e^{2x}} \qquad \text{and} \qquad \boxed{y_2(x) = e^{-2x}.} \tag{2}$$

Now, if y_1 and y_2 were not linearly independent, one would be a multiple of the other. That is, there would be a constant, α, such that,

$$e^{2x} = \alpha e^{-2x}.$$

But then, this would mean $\alpha = e^{4x}$ making it dependent upon x, and not constant. Thus, no such α exists, and the two solutions in equation (2) are linearly independent.

b) The general solution to equation (1) is a linear combination of the two linearly independent solutions in equations (2). That is,

$$\boxed{y(x) = Ay_1(x) + By_2(x) = Ae^{2x} + Be^{-2x},} \tag{3}$$

where A and B are constants (independent of x).

c) Since one of the boundary conditions is applied to y', we first differentiate equation (3) to get:

$$y'(x) = 2Ae^{2x} - 2Be^{-2x}. \qquad 4)$$

Then, setting $y(0) = 0$ and $y'(0) = 1$, we get:

$$y(0) = A + B = 0 \qquad \text{and} \qquad y'(0) = 2A - 2B = 1.$$

Solving these for A and B, we get:

$$A = \frac{1}{4} \qquad \text{and} \qquad B = -\frac{1}{4},$$

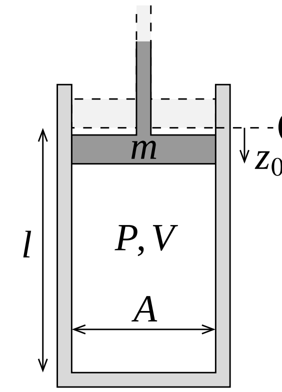
which, when substituted into equation (3), gives:

$$y(x) = \frac{1}{4}\left(e^{2x} - e^{-2x}\right) = \frac{1}{2}\underbrace{\frac{e^{2x} - e^{-2x}}{2}}_{\sinh 2x} \quad \Rightarrow \quad \boxed{y(x) = \tfrac{1}{2}\sinh 2x,}$$

as desired.

Problem 2 A piston of mass m and cross-sectional area A slides without friction and air leak in a cylinder of length l closed at one end. In the vertical position, it rests on top of an air column with equilibrium volume V and pressure P (the latter being the sum of the atmospheric pressure, P_{atm}, and the pressure exerted by the weight of the piston, mg/A).

From the equilibrium position, the piston is pushed into the cylinder by a distance $z_0 \ll l$, then released. The system oscillates rapidly enough that the gas may be considered *adiabatic*; heat neither enters nor leaves the gas as it repeatedly contracts and expands under the oscillation. For such a gas, the relationship between pressure and volume is given by the adiabatic gas law:



$$PV^\gamma = \kappa \quad \text{(constant)}, \tag{1}$$

where γ is the *ratio of specific heats* (5/3 for a monatomic gas, 7/5 for a diatomic gas, 4/3 for a polyatomic gas).

a) Make the approximation,

$$\delta P \approx \frac{dP}{dV}\,\delta V, \tag{2}$$

where $\delta P \ll P$ and $\delta V \ll V$, to show that for small displacements from equilibrium, z $(-z_0 \le z \le z_0)$, the unbalanced force is given by,

$$F = A\delta P = -\frac{\gamma P A}{l}\,z. \tag{3}$$

b) Show how equation (3) represents a simple harmonic oscillator with a period given by,

$$T = 2\pi\sqrt{\frac{ml}{\gamma P A}}.$$

c) Find a numerical value for the period of oscillation for $m = 1.00\,\text{kg}$, $l = 1.00\,\text{m}$, $A = 0.125\,\text{m}^2$, and assuming a diatomic gas such as the atmosphere (to a darn good approximation). For P_{atm}, use $1.013 \times 10^5\,\text{N}\,\text{m}^{-2}$ and for g, use $9.81\,\text{m}\,\text{s}^{-2}$.

Solution: a) From equation (1), we write,

$$P = \kappa V^{-\gamma} \quad \Rightarrow \quad \frac{dP}{dV} = -\gamma\kappa V^{-\gamma-1} = -\gamma\frac{\kappa V^{-\gamma}}{V} = -\gamma\frac{P}{V}.$$

Thus, equation (2) gives us:

$$\delta P = -\gamma\frac{P}{V}\,\delta V, \tag{4}$$

which means that a small volume change, $\delta V = Az$, results in a small restoring (because of the $-$ sign) pressure, δP. Defining the unbalanced force as $F = A\delta P$, equation (4) becomes,

$$F = -\gamma\frac{PA}{V}\,Az \quad \Rightarrow \quad \boxed{F = -\frac{\gamma P A}{l}\,z,}$$

since $V/A = l$. This is equation (3).

b) Since $\delta P \ll P$, $P \sim$ constant and equation (3) has the form of a Hooke's Law force, $F = -kx$. Thus, this system is a simple harmonic oscillator. Then, from Newton's 2$^{\text{nd}}$ law, $F = m\ddot{z}$ and we have:

$$\ddot{z} = -\frac{\gamma PA}{ml} z \quad \Rightarrow \quad \omega_0^2 = \frac{\gamma PA}{ml} \quad \Rightarrow \quad \boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{ml}{\gamma PA}},}$$

as desired.

c) For the values given,

$$P = P_{\text{atm}} + \frac{mg}{A} = 1.013 \times 10^5 + \frac{(1.00)(9.81)}{0.125} \sim 1.104 \times 10^5\,\text{N}\,\text{m}^{-2},$$

retaining four significant figures since this is an intermediate result. Thus,

$$T = 2\pi\sqrt{\frac{5}{7}\frac{(1.00)(1.00)}{(1.014 \times 10^5)(0.125)}} \sim \underline{\underline{4.72 \times 10^{-2}\,\text{s}}}$$

or about 21.2 Hz (oscillations per second). It is for this reason that demonstrating the "springiness of air" is difficult in a classroom setting.

Tutorial 3.2

Problem 3 (FC 3.5) A particle undergoing simple harmonic motion has a velocity $\dot{x}_1$ when the displacement is x_1, and a velocity $\dot{x}_2$ when the displacement is x_2. In terms of the given quantities, find the:

a) angular frequency, ω_0;

b) period, T; and

c) amplitude of the motion, x_0.

Hint: It may be easier if you use the SHO solution form: $x(t) = x_0\cos(\omega_0 t - \phi_0)$. Alternately, you could consider the mechanical energy of the oscillator.

Solution: a) Following the hint, start with the equations,

$$x(t) = x_0\cos(\omega_0 t - \phi_0) \quad \Rightarrow \quad \dot{x}(t) = -x_0\omega_0\sin(\omega_0 t - \phi_0),$$

and evaluate them at each of the two points. Thus,

$$x(t_1) = x_1 = x_0\cos(\omega_0 t_1 - \phi_0); \tag{1}$$

$$\dot{x}(t_1) \;=\; \dot{x}_1 \;=\; -x_0\omega_0\sin(\omega_0 t_1 - \phi_0); \tag{2}$$

$$x(t_2) \;=\; x_2 \;=\; x_0\cos(\omega_0 t_2 - \phi_0); \tag{3}$$

$$\dot{x}(t_2) \;=\; \dot{x}_2 \;=\; -x_0\omega_0\sin(\omega_0 t_2 - \phi_0), \tag{4}$$

where t_1 is the time at which $x(t) = x_1$ and $\dot{x}(t) = \dot{x}_1$; ditto for t_2. Now, the problem makes no mention of t_1 and t_2, so we must look to eliminate these variables. Since they appear exclusively in sine and cosine functions, the best way to do this is to figure out how we can get a $\cos^2 + \sin^2$ in the algebra.

To this end, divide Eq. (2) by ω_0, then square both this and Eq. (1) and add:

$$\begin{array}{rcl} x_1^2 &=& x_0^2\cos^2(\omega_0 t_1 - \phi_0) \\ + \left(\dfrac{\dot{x}_1^2}{\omega_0^2} \right. &=& \left. x_0^2\sin^2(\omega_0 t_1 - \phi_0) \right) \\ \hline x_1^2 + \dfrac{\dot{x}_1^2}{\omega_0^2} &=& x_0^2\big(\underbrace{\cos^2(\omega_0 t_1 - \phi_0) + \sin^2(\omega_0 t_1 - \phi_0)}_{1}\big) \;=\; x_0^2 \end{array} \tag{5}$$

Combining Eq. (3) and (4) in the same way, we get,

$$x_2^2 + \frac{\dot{x}_2^2}{\omega_0^2} \;=\; x_0^2. \tag{6}$$

Eliminating x_0^2 in Eq. (5) and (6) gives us,

$$x_1^2 + \frac{\dot{x}_1^2}{\omega_0^2} \;=\; x_2^2 + \frac{\dot{x}_2^2}{\omega_0^2} \quad\Rightarrow\quad \boxed{\omega_0 \;=\; \sqrt{\frac{\dot{x}_1^2 - \dot{x}_2^2}{x_2^2 - x_1^2}}\,,} \tag{7}$$

after a little algebra. Note the opposite order of the indices in the numerator and denominator.

b) From Eq. (7), the period follows trivially,

$$\boxed{T \;=\; \frac{2\pi}{\omega_0} \;=\; 2\pi\sqrt{\frac{x_2^2 - x_1^2}{\dot{x}_1^2 - \dot{x}_2^2}}\,.} \tag{8}$$

c) To find the amplitude, x_0, we back substitute Eq. (7) into either Eq. (5) or (6) (I'm choosing Eq. 5), to get,

$$x_0^2 \;=\; x_1^2 + \frac{\dot{x}_1^2}{\omega_0^2} \;=\; x_1^2 + \dot{x}_1^2\frac{x_2^2 - x_1^2}{\dot{x}_1^2 - \dot{x}_2^2} \;=\; \frac{\cancel{x_1^2\dot{x}_1^2} - x_1^2\dot{x}_2^2 + \dot{x}_1^2 x_2^2 - \cancel{\dot{x}_1^2 x_1^2}}{\dot{x}_1^2 - \dot{x}_2^2}$$

$$\Rightarrow \quad \boxed{x_0 \;=\; \sqrt{\frac{\dot{x}_1^2 x_2^2 - x_1^2\dot{x}_2^2}{\dot{x}_1^2 - \dot{x}_2^2}}\,.} \tag{9}$$

Alternately, we could use the mechanical energy of the oscillator. To wit, starting with equation (2.3.1) in the class notes:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant},$$

we can write,

$$E_1 = E_2 \quad \Rightarrow \quad \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}kx_2^2 \quad \Rightarrow \quad m(\dot{x}_1^2 - \dot{x}_2^2) + k(x_1^2 - x_2^2) = 0$$

$$\Rightarrow \quad \frac{k}{m} = \frac{\dot{x}_1^2 - \dot{x}_2^2}{x_2^2 - x_1^2} \quad \Rightarrow \quad \boxed{\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{\dot{x}_1^2 - \dot{x}_2^2}{x_2^2 - x_1^2}},}$$

identical to Eq. (7).

The period, T, is as before (Eq. 8). As for the amplitude, x_0, we note that at maximum extension when the particle is momentarily at rest,

$$E = \frac{1}{2}kx_0^2.$$

Since mechanical energy is conserved, we can equate this to either E_1 or E_2[1]. Choosing E_1,

$$\frac{1}{2}kx_0^2 = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}kx_1^2$$

$$\Rightarrow \quad x_0^2 = \frac{m}{k}\dot{x}_1^2 + x_1^2 = \frac{x_2^2 - x_1^2}{\dot{x}_1^2 - \dot{x}_2^2}\dot{x}_1^2 + x_1^2 = \frac{x_2^2\dot{x}_1^2 - \cancel{x_1^2\dot{x}_1^2} + \cancel{\dot{x}_1^2 x_1^2} - \dot{x}_2^2 x_1^2}{\dot{x}_1^2 - \dot{x}_2^2}$$

$$\Rightarrow \quad \boxed{x_0 = \sqrt{\frac{x_2^2\dot{x}_1^2 - \dot{x}_2^2 x_1^2}{\dot{x}_1^2 - \dot{x}_2^2}},}$$

identical to Eq. (9). Note that on interchanging the indices 1 and 2, we get:

$$\sqrt{\frac{x_1^2\dot{x}_2^2 - \dot{x}_1^2 x_2^2}{\dot{x}_2^2 - \dot{x}_1^2}} = \sqrt{\frac{\dot{x}_1^2 x_2^2 - x_1^2\dot{x}_2^2}{\dot{x}_1^2 - \dot{x}_2^2}} = x_0,$$

also identical to Eq. (9). Thus, we say the result is *symmetric under the interchange of indices*, as intuited in the footnote. The anticipation of a symmetry can act as a good check on the algebra. If the final result were not symmetric in the interchange of the indices, this would have told us we made an error.

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[1]It's worth noting that whether we use E_1 or E_2, the final result must be the same. Thus, our intuition is that the form of the final result shall be *symmetric* in the indices 1 and 2.