Solutions to Tutorial 4

PHYS 2302 (Mechanics I); D. A. Clarke

Tutorial 4.1

Problem 1

a) Show that Eq. (2.2.1), $\ddot{x} = -\omega_0^2 x$, can be derived from Eq. (2.3.1), $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 = \text{constant}$. *Hint*: Try differentiating the latter with respect to t.

b) In terms of E, what is the maximum speed of a SHO?

c) Use Eq. (2.3.3) to find the frequency of a simple pendulum.

Solution: a) Following the hint, differentiate Eq. (2.3.1) term by term to get,

$$\frac{dE}{dt} = \frac{d}{dt}\frac{m\dot{x}^2}{2} + \frac{d}{dt}\frac{m\omega_0^2 x^2}{2} \quad \Rightarrow \quad 0 = \frac{m2\dot{x}\ddot{x}}{2} + \frac{m\omega_0^2 2x\dot{x}}{2} = m\dot{x}(\ddot{x} + \omega_0^2 x)$$

$$\Rightarrow \quad \boxed{\ddot{x} = -\omega_0^2 x,}$$

which is Eq. (2.2.1), Hooke's Law.

b) Since E is constant, we may evaluate it at any point in the cycle of the oscillator. In particular, when $x = 0$ and $\dot{x} = v_{\max}$ is a maximum,

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_{\max}^2 \quad \Rightarrow \quad \boxed{v_{\max} = \sqrt{\frac{2E}{m}}.}$$

c) For a simple pendulum, we found in Example 2.5,

$$U(x) \approx \frac{mg}{2l}x^2,$$

where x is the horizontal displacement of the bob from its equilibrium (vertical) position. Taking its derivative twice, we get,

$$U'(x) = \frac{mg}{l}x \quad \Rightarrow \quad U''(x) = \frac{mg}{l},$$

a constant. Then using Eq. (2.3.3), we get,

$$\boxed{\omega_0^2 = \frac{U''(0)}{m} = \frac{g}{l},}$$

as expected.

Tutorial 4.2

Problem 2 (FC 3.9) Show that the ratio of two successive maxima in the displacement of an underdamped harmonic oscillator is constant.

Solution: Start with equation (2.4.13) in the class notes,

$$x(t) \;=\; Ae^{-\gamma t}\cos(\omega_{\mathrm{d}}t - \theta_0),$$

where $A = x_0\omega_0/\omega_{\mathrm{d}}$, $x_0 = x(0)$ is the position of the oscillator at $t = 0$, ω_0 is the frequency of the undamped oscillator, $\omega_{\mathrm{d}} = \sqrt{\omega_0^2 - \gamma^2}$ is the frequency of the damped oscillator, $\gamma = b/2m$ is the exponential damping factor, and θ_0 is the phase.

The time between successive maxima, say the i^{th} and $i+1^{\text{st}}$, is a period of oscillation, T_{d}, and thus,

$$t_{i+1} - t_i \;=\; T_{\mathrm{d}} \;=\; \frac{2\pi}{\omega_{\mathrm{d}}}. \tag{1}$$

On comparing the amplitudes at these times, we have:

$$\frac{x_{i+1}}{x_i} \;=\; \frac{\cancel{A}e^{-\gamma t_{i+1}}\cos(\omega_{\mathrm{d}}t_{i+1} - \theta_0)}{\cancel{A}e^{-\gamma t_i}\cos(\omega_{\mathrm{d}}t_i - \theta_0)} \;=\; e^{-\gamma(t_{i+1}-t_i)}\underbrace{\frac{\cos(\omega_{\mathrm{d}}t_i + 2\pi - \theta_0)}{\cos(\omega_{\mathrm{d}}t_i - \theta_0)}}_{1} \;=\; e^{-\gamma T_{\mathrm{d}}},$$

using equation (1). Thus, the ratio of successive maxima is given by,

$$\boxed{\frac{x_{i+1}}{x_i} \;=\; e^{-2\pi\gamma/\omega_{\mathrm{d}}},} \tag{2}$$

a constant, as desired. Note that equation (2) also gives the ratio of consecutive *minima*.

Problem 3 An underdamped harmonic oscillator consists of a spring with spring constant $k = 10.0\,\mathrm{N\,m^{-1}}$ and a mass $m = 0.100\,\mathrm{kg}$. The mass is displaced from equilibrium by 4.00 cm and it is observed that after oscillating for 10 s, an integral number of oscillations have occurred where the amplitude is now just 2.00 cm.

a) What is the damping coefficient, γ?

b) What is the natural oscillation frequency, ω_{d}, and thus the period, T_{d}, of the damped oscillator?

Solution: a) For an underdamped oscillator, we start with Eq. (2.4.13) in the class notes,

$$x(t) \;=\; x_0 e^{-\gamma t}\frac{\omega_0}{\omega_{\mathrm{d}}}\cos(\omega_{\mathrm{d}}t - \theta_0),$$

where the amplitude at $t = 0$ is:

$$A(0) \;=\; x_0\frac{\omega_0}{\omega_{\mathrm{d}}}\cos(\theta_0) \;=\; x_0,$$

using the second of Eq. (2.4.12) in the class notes. Then, after a time t and an integral number of oscillations [and thus $(\omega_0/\omega_{\mathrm{d}})\cos(\omega_{\mathrm{d}}t - \theta_0) = 1$], the amplitude is:

$$A(t) \;=\; x_0 e^{-\gamma t}.$$

Thus, if $A(0) = 4.00\,\mathrm{cm}$ and $A(10\,\mathrm{s}) = 2.00\,\mathrm{cm}$,

$$\frac{A(10)}{A(0)} \;=\; \frac{\cancel{x_0}e^{-10\gamma}}{\cancel{x_0}} \;=\; e^{-10\gamma} \;=\; \frac{2.00}{4.00} \;=\; \frac{1}{2}$$

$$\Rightarrow \quad 10\gamma \;=\; \ln 2 \quad \Rightarrow \quad \gamma \;=\; \frac{\ln 2}{10} \;\sim\; \underline{\underline{0.0693}}.$$

b) From the class notes, we have:

$$\omega_{\mathrm{d}} \;=\; \sqrt{\omega_0^2 - \gamma^2} \;=\; \sqrt{\frac{k}{m} - \frac{(\ln 2)^2}{100}} \;\sim\; 9.9998 \;\sim\; \underline{\underline{10.0\,\mathrm{rad\,s^{-1}}}},$$

to three significant figures. Thus,

$$T_{\mathrm{d}} \;=\; \frac{2\pi}{\omega_{\mathrm{d}}} \;\sim\; \underline{\underline{0.628\,\mathrm{s}}}.$$

Tutorial 4.3

Problem 4 For the underdamped harmonic oscillator described in the last tutorial, we found the damping coefficient to be $\gamma = \frac{1}{10}\ln 2$ and the natural oscillation frequency to be $\omega_{\mathrm{d}} \sim 10.0\,\mathrm{rad\,s^{-1}}$.

a) What is the phase offset of the oscillator, θ_0?

b) What is the quality factor, Q_{d}?

c) Approximately how many complete oscillations has the oscillator undergone by the time the amplitude falls to $x = x_0/2 = 2.00\,\mathrm{cm}$?

Solution: a) From the first of equations (2.4.12), the phase offset is given by:

$$\theta_0 \;=\; \tan^{-1}\frac{\gamma}{\omega_\mathrm{d}} \;\sim\; \tan^{-1}\frac{\ln 2}{100.0} \;\sim\; \tan^{-1}(6.931\times 10^{-3}) \;\sim\; \underline{\underline{0.00693\,\mathrm{rad} \;\sim\; 0.397^\circ}}.$$

b) From equation (2.4.16) in the class notes, the quality factor is given by:

$$Q_\mathrm{d} \;=\; \frac{\omega_\mathrm{d}}{2\gamma} \;\sim\; \frac{100.0}{2\ln 2} \;\sim\; \underline{\underline{72.1}}.$$

c) From Example 2.9 in the class notes, the relation between the quality factor and the number of complete oscillations, n, before the amplitude of oscillation falls off by a factor of two (the "half time" or "half-life") is:

$$Q_\mathrm{d} \;=\; \frac{\pi n}{\ln 2} \quad\Rightarrow\quad n \;=\; Q_\mathrm{d}\frac{\ln 2}{\pi} \;=\; \frac{100.0}{2\cancel{\ln 2}}\frac{\cancel{\ln 2}}{\pi} \;=\; \frac{100.0}{2\pi} \;\sim\; \underline{\underline{15.9}}.$$

Thus, the oscillator undergoes about 16 complete oscillations by the time its amplitude falls off by a factor of two.

Problem 5 Solve by "inspection", "direct integration", and/or "trial exponentials" the linear, second-order, inhomogeneous ODE:

$$y''(x) + 4y(x) \;=\; f(x), \tag{1}$$

for boundary conditions $y(0) = -1$ and $y'(0) = 1$, where:

a) $f(x) = -2$;

b) $f(x) = 4e^{2x}$.

As discussed in §2.5 of the course notes, to solve such an *inhomogeneous* $[f(x) \neq 0]$ equation with boundary conditions, you must:

1. find two linearly independent solutions to the *homogeneous* equation [with $f(x) = 0$]; call these $y_1(x)$ and $y_2(x)$;
2. construct the general solution to the homogeneous equation,

   $$y_\mathrm{h}(x) \;=\; Ay_1(x) + By_2(x),$$

   where A and B are constants;
3. by inspection or trial exponentials, find a *particular* solution, $y_\mathrm{p}(x)$ [*anything* that solves equation (1)];

4. write down the general solution to equation (1),

$$y(x) \;=\; y_{\rm h}(x) + y_{\rm p}(x);$$

5. and finally, apply boundary conditions to evaluate A and B.

Solution: For each part, a and b, we first solve the homogeneous version of equation (1),

$$y_{\rm h}''(x) + 4y_{\rm h}(x) \;=\; 0, \quad \Rightarrow \quad y_{\rm h}''(x) \;=\; -4y_{\rm h}(x) \tag{2}$$

by "inspection". Thus, we ask "What functions have as their second derivatives -4 times themselves?" As argued in §2.2 in the class notes, two linearly independent solutions to equation (2) with this property are:

$$y_1(x) \;=\; \cos 2x \qquad \text{and} \qquad y_2(x) \;=\; \sin 2x,$$

from which we construct the general solution to (the homogeneous) equation (2):

$$y_{\rm h}(x) \;=\; A\cos 2x + B\sin 2x, \tag{3}$$

where A and B are constants to be determined from boundary conditions.

a) For $f(x) = -2$, equation (1) becomes:

$$y''(x) + 4y(x) \;=\; -2, \tag{4}$$

for which we search for *any* solution by inspection. Since all coefficients are constant, try $y_{\rm p} =$constant $\Rightarrow$ $y_{\rm p}'' = 0$, and equation (4) becomes:

$$4y_{\rm p} \;=\; -2 \quad \Rightarrow \quad y_{\rm p} \;=\; -\frac{1}{2}.$$

The general solution to equation (1) is thus,

$$y(x) \;=\; y_{\rm h}(x) + y_{\rm p} \;=\; A\cos 2x + B\sin 2x - \frac{1}{2} \quad \Rightarrow \quad y'(x) \;=\; -2A\sin 2x + 2B\cos 2x.$$

Applying the given boundary conditions, we get:

$$y(0) \;=\; A - \frac{1}{2} \;=\; -1 \quad \Rightarrow \quad A \;=\; -\frac{1}{2} \qquad \text{and} \qquad y'(0) \;=\; 2B \;=\; 1 \quad \Rightarrow \quad B \;=\; \frac{1}{2}.$$

Thus, the specific solution to equation (1) for the given boundary conditions and $f(x) = 2$ is:

$$\boxed{y(x) \;=\; \frac{1}{2}\big(\sin 2x - \cos 2x - 1\big) \;=\; (\sin x - \cos x)\cos x,} \tag{5}$$

after a little trig.

b) For $f(x) = 4e^{2x}$, equation (1) becomes:

$$y''(x) + 4y(x) \;=\; 4e^{2x}, \tag{6}$$

for which we search for *any* solution by trial exponentials. Noting that if $y_{\rm p} \propto e^{2x}$, all x-dependence cancels out, we are motivated to try $y_{\rm p}(x) = Ce^{2x}$, in which case equation (6) becomes:

$$4C\cancel{e^{2x}} + 4C\cancel{e^{2x}} \;=\; 4\cancel{e^{2x}} \quad\Rightarrow\quad 8C \;=\; 4 \quad\Rightarrow\quad C \;=\; \frac{1}{2}.$$

The general solution to equation (1) is thus,

$$y(x) \;=\; y_{\rm h}(x) + y_{\rm p}(x) \;=\; A\cos 2x + B\sin 2x + \frac{1}{2}e^{2x}$$

$$\Rightarrow\quad y'(x) \;=\; -2A\sin 2x + 2B\cos 2x + e^{2x}.$$

Applying the given boundary conditions, we get:

$$y(0) \;=\; A + \frac{1}{2} \;=\; -1 \quad\Rightarrow\quad A \;=\; -\frac{3}{2} \qquad \text{and} \qquad y'(0) \;=\; 2B + 1 \;=\; 1 \quad\Rightarrow\quad B \;=\; 0.$$

Thus, the specific solution to equation (1) for the given boundary conditions and $f(x) = 4e^{2x}$ is:

$$\boxed{y(x) \;=\; -\frac{3}{2}\cos 2x + \frac{1}{2}e^{2x}.} \tag{7}$$

Note how different the solutions in equations (5) and (7) look; all because of the different inhomogeneous function $f(x)$ on the RHS of equation (1).

As an exercise, you should check to make sure that equations (5) and (7) do indeed solve equation (1) for the respective $f(x)$, and that in each case, $y(0) = -1$ and $y'(0) = 1$.
