Solutions to Tutorial 5

PHYS 2302 (Mechanics I); D. A. Clarke

Tutorial 5.1

Problem 1 Show that,

$$\frac{\cos\phi}{\cos\theta} = \frac{\omega_0}{\omega_{\rm d}},$$

where θ is the "phase lag" of transient term, ϕ is the "phase lag" of steady state term, and where ω_0 and $\omega_{\rm d}$ are, respectively, the natural undamped and damped oscillation frequencies.

Solution: Referring to the right triangle from class that defined θ, the transient phase lag, we can write,

$$\cos\theta = \frac{\omega_{\rm d}}{\sqrt{\omega_{\rm d}^2 + (\gamma + \omega\tan\phi)^2}}.$$

Now,

$$\begin{aligned}
\omega_{\rm d}^2 + (\gamma + \omega\tan\phi)^2 &= \underbrace{\omega_{\rm d}^2 + \gamma^2}_{\omega_0^2} + \underbrace{2\gamma\omega\tan\phi + \omega^2\tan^2\phi}_{\tan\phi = 2\gamma\omega/(\omega_0^2-\omega^2)} \\
&= \omega_0^2 + \frac{4\gamma^2\omega^2}{\omega_0^2 - \omega^2} + \frac{4\gamma^2\omega^4}{(\omega_0^2-\omega^2)^2} \\
&= \omega_0^2 + \frac{4\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2}(\omega_0^2 - \cancel{\omega^2} + \cancel{\omega^2}) \\
&= \omega_0^2\left(1 + \frac{4\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2}\right) = \omega_0^2\underbrace{\left(\frac{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}{(\omega_0^2-\omega^2)^2}\right)}_{\chi^2 \text{ Eq. (2.6.6)}} \\
&= \frac{\omega_0^2}{\cos^2\phi}.
\end{aligned}$$

$$\Rightarrow \quad \cos\theta = \frac{\omega_{\rm d}}{\sqrt{\omega_0^2/\cos^2\phi}} = \frac{\omega_{\rm d}}{\omega_0}\cos\phi \quad \Rightarrow \quad \boxed{\frac{\cos\phi}{\cos\theta} = \frac{\omega_0}{\omega_{\rm d}}},$$

as desired.

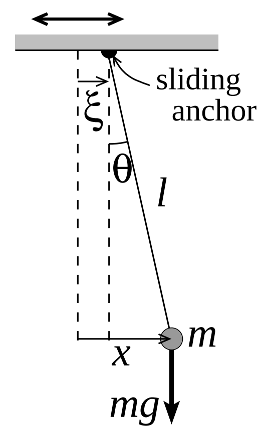
Problem 2 A simple pendulum of length l and bob mass m swings back and forth with an effective weak damping coefficient, γ. The anchor can be driven to slide back and forth horizontally in simple harmonic motion with a maximum amplitude ξ_0.

a) If the horizontal position of the bob is x and the horizontal position of the slider is ξ, show that for small oscillations, the differential equation of motion for the bob is:

$$\ddot{x} + 2\gamma\dot{x} + \frac{g}{l}x = \frac{g}{l}\xi.$$

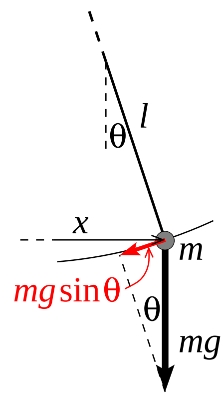
b) If $\xi(t) = \xi_0 \cos\omega t$, find expressions for the phase and amplitude of the steady-state oscillations in terms of ω_0, ω, ξ_0, and γ.

c) If it takes 50 swings for the amplitude to fall by a factor of $e = 2.71828\ldots$ when the pendulum is swinging freely ($\xi_0 = 0$), find an expression for γ in terms of ω_0. Is the assumption of weak damping justified?



Solution: a) As seen in the inset, the restoring force is given by $-mg\sin\theta$ while the damping force is given by $-bv = -2m\gamma\dot{x}$ (since $b = 2\gamma m$). Thus, from Newton's 2^{nd} law we have,

$$\sum F = -mg\sin\theta - 2\gamma m\dot{x} = ma = m\ddot{x}. \tag{1}$$



Next, from the diagram in the problem description,

$$\sin\theta = \frac{x - \xi}{l},$$

and equation (1) becomes:

$$-mg\frac{x-\xi}{l} - 2\gamma m\dot{x} = m\ddot{x} \quad \Rightarrow \quad \boxed{\ddot{x} + 2\gamma\dot{x} + \frac{g}{l}x = \frac{g}{l}\xi,} \tag{2}$$

as desired.

b) For $\xi(t) = \xi_0 \cos\omega t$, equation (2) becomes:

$$\ddot{x} + 2\gamma\dot{x} + \frac{g}{l}x = \frac{g}{l}\xi_0\cos\omega t,$$

which has exactly the same form as equation (2.6.1) from the class notes for a driven, damped oscillator with F_0/m replaced by $g\xi_0/l$. Therefore, from equations (2.6.4) and (2.6.7) in the class notes, we can immediately write down,

$$\boxed{\tan\phi = \frac{2\omega\gamma}{\omega_0^2 - \omega^2}} \quad \text{and} \quad \boxed{A = \frac{\omega_0^2\xi_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\gamma^2}},} \tag{3}$$

where $\omega_0^2 = g/l$ is the oscillation frequency of the undamped pendulum.

c) For a weakly damped system, $\omega_d \approx \omega_0 = \sqrt{g/l}$, Thus, the time for 50 periods is:

$$t_{50} = 50T = 50\frac{2\pi}{\omega_0}.$$

Then, if after 50 oscillations, the amplitude has fallen by a factor of e,

$$e^{-\gamma t_{50}} = \frac{1}{e} = e^{-1} \quad \Rightarrow \quad \gamma t_{50} = 1 \quad \Rightarrow \quad \boxed{\gamma = \frac{1}{t_{50}} = \frac{\omega_0}{100\pi}.}$$

Since $\gamma/\omega_0 = 1/(100\pi) \ll 1$, the assumption of weak damping is justified.

Tutorial 5.2

Problem 3 (FC 3.11) A mass m moves along the x-axis subject to a restoring force $F_r = -\frac{17}{2}\beta^2 mx$ and a drag force $F_d = -3\beta m\dot{x}$, where x is the distance from the origin and β is a constant. In addition, a driving force, $F = mA\cos\omega t$ is applied to the particle along the x-axis, where A is another constant.

a) What value of ω results in steady-state oscillations about the origin ($x = 0$) with maximum amplitude?

b) For the value of ω found in part a, what is the maximum amplitude?

Solution: For the problem described, Newton's second law gives:

$$\sum F = F_r + F_d + F = -\frac{17}{2}\beta^2 mx - 3\beta m\dot{x} + mA\cos\omega t = m\ddot{x}$$

$$\Rightarrow \quad \ddot{x} + 3\beta\dot{x} + \frac{17}{2}\beta^2 x = A\cos\omega t,$$

is the differential equation of motion for the driven, damped oscillator. Comparing this to the generic ODE (equation 2.6.1 from the class notes),

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t,$$

we must have,

$$\gamma = \frac{3}{2}\beta, \qquad \omega_0^2 = \frac{17}{2}\beta^2, \qquad \text{and} \qquad F_0 = mA.$$

a) Here, we're looking for the resonant frequency given by equation (2.6.12) in the class notes,

$$\omega_r^2 = \omega_0^2 - 2\gamma^2 = \frac{17}{2}\beta^2 - 2\left(\frac{3}{2}\beta\right)^2 = 4\beta^2 \quad \Rightarrow \quad \boxed{\omega_r = 2\beta.}$$

b) Start by noting that,

$$\omega_{\rm d}^2 = \omega_0^2 - \gamma^2 = \frac{17}{2}\beta^2 - \left(\frac{3}{2}\beta\right)^2 = \frac{25}{4}\beta^2 \quad\Rightarrow\quad \omega_{\rm d} = \frac{5}{2}\beta.$$

Thus, the resonant amplitude is given by equation (2.6.13) from the class notes:

$$A_{\rm max} = \frac{F_0}{b\,\omega_{\rm d}} = \frac{F_0}{2m\gamma\omega_{\rm d}} = \frac{\cancel{m}A}{\cancel{2}\cancel{m}\,\frac{3}{\cancel{2}}\beta\,\frac{5}{2}\beta} \quad\Rightarrow\quad \boxed{A_{\rm max} = \frac{2A}{15\beta^2}.}$$

Problem 4 (FC 3.14) An oscillator consists of a spring with spring constant, k, and a damping coefficient, γ, equal to one half the critical value. If the undamped frequency of the oscillator is ω_0 and it is driven by a force $F_0\cos(2\omega_0 t)$, find:

a) the resonant frequency, $\omega_{\rm r}$;

b) the phase, ϕ, between the driving force and the oscillator displacement; and

c) the steady state amplitude.

Solution: a) From equation (2.6.12) in the class notes, the resonant frequency is given by:

$$\omega_{\rm r} = \sqrt{\omega_0^2 - 2\gamma^2}. \tag{1}$$

Critical damping occurs for $\gamma_{\rm cd} = \omega_0$. Thus, for $\gamma = \frac{1}{2}\gamma_{\rm cd} = \frac{1}{2}\omega_0$, equation (1) becomes:

$$\omega_{\rm r} = \sqrt{\omega_0^2 - 2\left(\frac{\omega_0}{2}\right)^2} \quad\Rightarrow\quad \boxed{\omega_{\rm r} = \frac{\omega_0}{\sqrt{2}} \sim 0.707\omega_0.}$$

b) The phase is given by any of equations (2.6.4) and (2.6.5) in the class notes. Choosing the first, and noting that the oscillator is driven at $\omega = 2\omega_0$, we have,

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2} = \frac{2(\frac{1}{2}\omega_0)(2\omega_0)}{\omega_0^2 - (2\omega_0)^2} = \frac{2\cancel{\omega_0^2}}{-3\cancel{\omega_0^2}} = -\frac{2}{3}$$

$$\Rightarrow \quad \boxed{\phi = \tan^{-1}\left(-\frac{2}{3}\right) \sim 2.55\,\text{rad} \sim 146.^\circ,}$$

choosing ϕ so that $0 < \phi < \pi$. What this means is the driver leads the oscillator by $2.55/2\pi \sim 41\%$ of a period. As with any applied force, it takes time for an force to stop, then reverse the direction of a moving mass, and thus the applied force and motion are typically "out of phase".

c) For the steady-state amplitude, we use equation (2.6.7) from the class notes:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} = \frac{F_0\omega_0^2/k}{\sqrt{\left(\omega_0^2 - 4\omega_0^2\right)^2 + 4\left(\frac{1}{2}\omega_0\right)^2\left(2\omega_0\right)^2}}$$

$$\Rightarrow \quad \boxed{A = \frac{F_0}{\sqrt{13}\,k} \sim 0.277\frac{F_0}{k}.}$$
