Solutions to Tutorial 6

PHYS 2302 (Mechanics I); D. A. Clarke

Tutorial 6.1

Problem 1 (FC 4.1) Find the forces corresponding to each of the following potential energy functions.

a) $U(x,y,z) \;=\; cxyz + C$

b) $U(x,y,z) \;=\; \alpha x^2 + \beta y^2 + \gamma z^2 + C$

c) $U(x,y,z) \;=\; ce^{-(\alpha x+\beta y+\gamma z)}$

d) $U(r) \;=\; cr^n$ in spherical coordinates

Solution: The relevant equation here is: $\vec{F}(\vec{r}) \;=\; -\nabla U(\vec{r})$.

a) Thus, for $U(x,y,z) = cxyz + C$,

$$\vec{F} \;=\; -\nabla(cxyz + C) \;=\; -c\big(\partial_x(xyz), \partial_y(xyz), \partial_z(xyz)\big) \;=\; -c(yz, zx, xy).$$

b) For $U(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2 + C$,

$$\vec{F} \;=\; -\nabla(\alpha x^2 + \beta y^2 + \gamma z^2 + C) \;=\; -2(\alpha x, \beta y, \gamma z).$$

c) For $U(x,y,z) = ce^{-(\alpha x+\beta y+\gamma z)}$,

$$\vec{F} \;=\; -\nabla(ce^{-(\alpha x+\beta y+\gamma z)}) \;=\; c(\alpha, \beta, \gamma)e^{-(\alpha x+\beta y+\gamma z)}.$$

d) For $U(r) = cr^n$ in spherical coordinates,

$$\vec{F} \;=\; -\nabla(ce^r) \;=\; -c\left(\partial_r, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\phi\right) r^n \;=\; -c(nr^{n-1}, 0, 0) \;=\; -cnr^{n-1}\hat{e}_r.$$

Problem 2 Which of the forces in parts a and b is/are conservative, and in part c, find the value of α that makes the force conservative.

a) $\vec{F}(x,y,z) \;=\; \big(xy + z^2, y(2z+x), 2xz + y^2\big)$

b) $\vec{F}(x,y,z) \;=\; (\sin x\, e^z)\hat{\imath} + (z \sin y)\hat{\jmath} - (\cos x\, e^z + \cos y)\hat{k}$

c) $\vec{F}(x,y,z) \;=\; \big(\cosh z - x^2 e^{\alpha y}, -x^3 e^{\alpha y} + 2yz, x \sinh z + y^2\big)$

Solution:

a) $$\nabla \times \vec{F} \;=\; (\partial_y F_z - \partial_z F - y, \partial_z F_x - \partial_x F_z, \partial_x F_y - \partial_y F_x)$$
$$= \; (2y - 2y, 2z - 2z, -y - x) \;\neq\; \vec{0},$$

and $\vec{F}$ is not conservative.

b) $\nabla \times \vec{F} \;=\; \big(\sin y - \sin y, e^z \sin x - \sin x\, e^z, 0 - 0\big) \;=\; \vec{0},$

and $\vec{F}$ is conservative

c) $\nabla \times \vec{F} \;=\; \big(2y - 2y, \sinh z - \sinh z, -3x^2 e^{\alpha y} + \alpha x^2 e^{\alpha y}\big) \;=\; \vec{0}$ provided $\alpha = 3$.

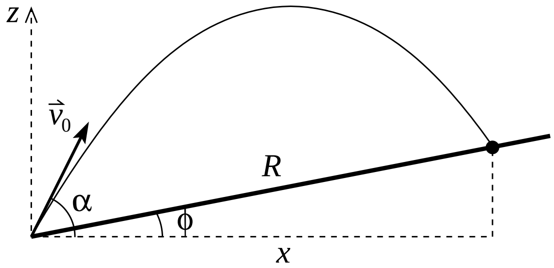
Thus, $\vec{F}$ is conservative if and only if $\alpha = 3$.

Tutorial 6.2

Problem 3 (FC 4.8) A gun is located at the bottom of a hill with constant inclination angle ϕ and aimed at a target up the hill.

a) If air resistance is negligible and α is the angle of elevation of the gun measured between the x-axis and $\vec{v}_0$, show that the range of the gun, R, up the hill is:

$$R \;=\; \frac{2v_0^2 \cos\alpha \sin(\alpha - \phi)}{g \cos^2\phi}.$$



b) What is the optimal angle of elevation, α, that maximises the range of the gun, and what is that maximum range?

Solution: a) The trajectory of the projectile is given by equation 4.3.9 (ed. 7):

$$z_{\rm p}(x) \;=\; x \tan\alpha - \frac{g}{2v_0^2 \cos^2\alpha} x^2,$$

where v_0 is the muzzle velocity. The equation describing the hill is: $z_{\rm h}(x) = x \tan\phi$.

To find point where the projectile hits the hill, equate $z_{\mathrm{p}}(x) = z_{\mathrm{h}}(x)$, and solve for x. Thus,

$$x\tan\phi \;=\; x\tan\alpha - \frac{g}{2v_0^2\cos^2\!\alpha}x^2 \quad\Rightarrow\quad x\left(\tan\alpha - \tan\phi - \frac{g}{2v_0^2\cos^2\!\alpha}x\right) \;=\; 0$$

$$\Rightarrow\quad x \;=\; 0 \quad \text{(origin)} \qquad \text{or} \qquad x \;=\; \frac{2v_0^2\cos^2\alpha}{g}(\tan\alpha - \tan\phi) \quad \text{(target)}$$

$$\Rightarrow\quad R \;=\; \frac{x}{\cos\phi} \;=\; \frac{2v_0^2\cos^2\!\alpha}{g\cos\phi}\left(\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\phi}{\cos\phi}\right) \;=\; \frac{2v_0^2\cos^{\cancel{2}}\!\alpha}{g\cos\phi}\,\frac{\sin\alpha\cos\phi - \sin\phi\cos\alpha}{\cancel{\cos\alpha}\cos\phi}$$

$$\Rightarrow\quad \boxed{R \;=\; \frac{2v_0^2\cos\alpha\sin(\alpha-\phi)}{g\cos^2\!\phi},} \tag{1}$$

as desired, using the trig identity $\sin(A-B) = \sin A\cos B - \sin B\cos A$.

b) To find the maximum range, set $\partial_\alpha R = 0$:

$$\partial_\alpha R \;=\; \frac{2v_0^2}{g\cos^2\!\phi}\big(-\sin\alpha\sin(\alpha-\phi) + \cos\alpha\cos(\alpha-\phi)\big) \;=\; 0$$

$$\Rightarrow\quad \cos\alpha\cos(\alpha-\phi) - \sin\alpha\sin(\alpha-\phi) \;=\; \cos(\alpha+\alpha-\phi) \;=\; 0,$$

using the trig identity: $\cos(A+B) = \cos A\cos B - \sin A\sin B$. Thus,

$$2\alpha - \phi \;=\; \frac{\pi}{2} \quad\Rightarrow\quad \boxed{\alpha \;=\; \frac{\phi}{2} + \frac{\pi}{4}.} \tag{2}$$

Substitute Eq. (2) into Eq. (1), and we get:

$$R_{\max} \;=\; \frac{2v_0^2\cos\left(\frac{\phi}{2}+\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}{g\cos^2\!\phi} \;=\; \frac{2v_0^2\frac{1}{\sqrt{2}}\big(\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\big)\frac{1}{\sqrt{2}}\big(\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\big)}{g\cos^2\!\phi}.$$

$$=\; \frac{v_0^2\big(\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\big)^2}{g\cos^2\!\phi} \;=\; \frac{v_0^2\big(1 - 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\big)}{g(1-\sin^2\!\phi)} \;=\; \frac{v_0^2\cancel{(1-\sin\phi)}}{g(1+\sin\phi)\cancel{(1-\sin\phi)}}$$

$$\Rightarrow\quad \boxed{R_{\max} \;=\; \frac{v_0^2}{g(1+\sin\phi)},}$$

using all sorts of trig identities! In particular, note that three lines up, $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ was used.

Maan Hani (class of '13) offered a very elegant solution to the problem of finding $R_{\max}$ and the optimal α when this problem was given on a midterm exam:

From the trig identity $2\sin a\cos b = \sin(a+b) + \sin(a-b)$, we have $2\sin(\alpha - \phi)\cos\alpha = \sin(2\alpha - \phi) - \sin\phi$, and equation (1) becomes:

$$R \;=\; \frac{v_0^2}{g\cos^2\!\phi}\big(\sin(2\alpha-\phi) - \sin\phi\big).$$

Since ϕ is set and since α is now isolated in a single trig function, we can see *by inspection* that R is maximised when $\sin(2\alpha - \phi) = 1$. Thus, the maximum range is given by:

$$R_{\max} \;=\; \frac{v_0^2}{g\cos^2\!\phi}(1-\sin\phi) \;=\; \frac{v_0^2}{g(1-\sin^2\!\phi)}(1-\sin\phi) \;=\; \frac{v_0^2}{g(1+\sin\phi)},$$

as before. Further, the optimal value for α is given by:

$$\sin(2\alpha-\phi) \;=\; 1 \quad\Rightarrow\quad 2\alpha - \phi \;=\; \frac{\pi}{2} \quad\Rightarrow\quad \alpha \;=\; \frac{\phi}{2} + \frac{\pi}{4}.$$

Tutorial 6.3

Problem 4 (FC 4.14) A projectile of mass m moves within the z-x plane, with z the vertical direction.

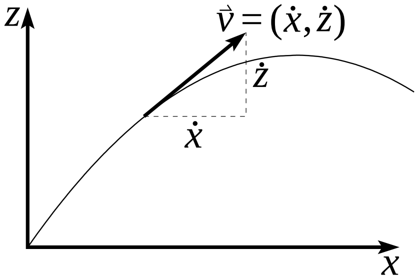
a) Write down the components of the differential equation of motion for the projectile when the air resistance is given by $\vec{F}_{\text{drag}} = -c_2 v^2 \hat{v}$, where c_2 is a constant, v is the speed of the projectile, and $\hat{v}$ is a unit vector in the direction of the velocity (thus, $\vec{v} = v\hat{v}$).

b) Are the equations separated? Explain.

c) Show that the x component of the velocity is given by: $\dot{x} = \dot{x}_0 e^{-\gamma s}$, where $\gamma = c_2/m$ and s is the distance the projectile has moved along its path.

Solution: a) The net force on the projectile comes from both gravity and air resistance. Since the velocity of the projectile is given by $\vec{v} = v\hat{v} = \dot{x}\hat{\imath} + \dot{z}\hat{k}$, we have:

$$\begin{aligned}\vec{F} \;&=\; -c_2 v^2\hat{v} - mg\hat{k} \;=\; -c_2 v\vec{v} - mg\hat{k}\\ &=\; -c_2 v\dot{x}\,\hat{\imath} - (c_2 v\dot{z} + mg)\hat{k},\end{aligned}$$



where c_2 is constant. Thus, the components of the differential equations of motion are:

$$F_x \;=\; m\ddot{x} \;=\; -c_2\sqrt{\dot{x}^2+\dot{z}^2}\,\dot{x} \qquad (1)$$

$$F_z \;=\; m\ddot{z} \;=\; -c_2\sqrt{\dot{x}^2+\dot{z}^2}\,\dot{z} - mg \qquad (2)$$

b) Evidently, these equations are not separated; $\dot{z}$ remains in the x-component and $\dot{x}$ remains in the z-component.

c) From (1), we have:

$$m\ddot{x} = m\frac{d\dot{x}}{dt} = -c_2 v\dot{x} \quad \Rightarrow \quad \frac{d\dot{x}}{\dot{x}} = -\frac{c_2}{m}vdt \quad \Rightarrow \quad \int_{\dot{x}_0}^{\dot{x}} \frac{d\dot{x}'}{\dot{x}'} = -\gamma\int_0^t vdt',$$

where $\gamma \equiv c_2/m$. Now, $vdt' = ds'$, an increment in displacement along the trajectory. Thus,

$$\ln \dot{x}'\Big|_{\dot{x}_0}^{\dot{x}} = -\gamma\int_0^s ds' \quad \Rightarrow \quad \ln\left(\frac{\dot{x}}{\dot{x}_0}\right) = -\gamma(s-0) \quad \Rightarrow \quad \boxed{\dot{x}(s) = \dot{x}_0 e^{-\gamma s}.}$$