Solutions to Tutorial 8

PHYS 2302 (Mechanics I); D. A. Clarke

Tutorial 8.1

Problem 1 (FC 5.1) A physics student who weighs 640 N (~140 lb.) stands on a bathroom spring scale in a moving elevator. What is the weight indicated on the scale if the elevator is:

a) accelerating upward with acceleration $g/4$;

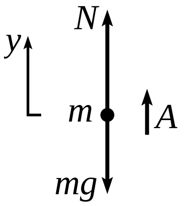
b) accelerating downward with acceleration $g/4$;

c) moving upward at constant speed;

d) moving downward at constant speed?

Solution: a) A bathroom scale doesn't actually measure *weight*, at least not directly. Instead, it responds to the *normal* force exerted on it and thus, by Newton's third law, the normal force the scale exerts on the person standing on it. You can quickly verify this by picking up the scale and squeezing it. The reading you get is not the weight of your thumb! It's a measure of the normal force your thumb exerts on the scale. It is only because we normally use such a scale on a horizontal plane standing vertically where $N = mg$ (*not* an action-reaction pair, remember!!) that we attribute this normal force to our weight.

Treating this problem from an *inertial* frame of reference as we are used to, from the FBD we see that:



$$\sum F = N - mg = mA = m\frac{g}{4} \quad\Rightarrow\quad N = \frac{5}{4}mg = 800\text{ N}.$$

Thus, $\boxed{\text{the bathroom scale reads 800 N}}$ as the elevator accelerates upward.

Viewing this problem instead from the accelerating frame of reference of the elevator (as the problem intends), use equation 5.14b of the text (p. 185 in ed. 7):

$$\sum\vec{F} - m\vec{A} = m\vec{a}\,', \tag{1}$$

where: $\vec{F}$ are the real forces;
$\vec{A}$ is the acceleration of observer's frame with respect to an inertial observer; and
$\vec{a}\,'$ is the acceleration of object under study with respect to observer's frame.

Here, $\vec{F} = N\hat{y} - mg\hat{y}$, $\vec{A} = \frac{1}{4}g\hat{y}$, and $\vec{a}\,' = 0$ (scale not accelerating with respect to observer standing on it). Thus, equation (1) becomes:

$$N - mg - m\frac{g}{4} = 0 \quad \Rightarrow \quad N = \frac{5}{4}mg,$$

as before.

Clearly this problem can be “intuited”, as I’m sure most of you did. However, you should study how I’ve taken care to present this solution properly, both from the perspective of an inertial observer and the accelerating observer, so that you can apply these ideas with greater confidence in problems in which your intuition will surely fail!

b) For $\vec{A} = -\frac{1}{4}g\hat{y}$, equation (1) becomes:

$$N - mg + m\frac{g}{4} = 0 \quad \Rightarrow \quad N = \frac{3}{4}mg = 480\text{ N}.$$

Thus, $\boxed{\text{the bathroom scale reads 480 N}}$ as the elevator accelerates downward.

c, d) With the elevator not accelerating, $\vec{A} = 0$. Since $\vec{a}\,'$ is still zero, equation (1) becomes:

$$N - mg = 0 \quad \Rightarrow \quad N = mg = 640\text{ N}.$$

Thus, $\boxed{\text{the bathroom scale reads 640 N,}}$ what the student considers to be his normal weight, when the elevator moves up or down at constant speed.

Problem 2 (FC 5.5) A truck driving on a level road suddenly slows with an acceleration $g/2$ causing a box of mass m in the rear to slide forward. If $\mu_\text{k} = 1/3$ between the box and floor, find the acceleration of the box a) relative to the truck, and b) relative to the road.

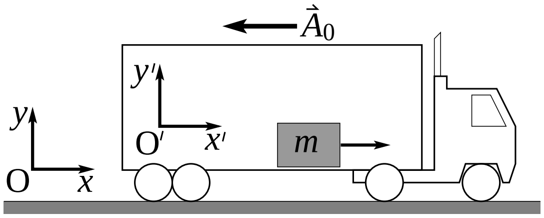
Solution: a) From the truck’s frame of reference (O′), Newton’s 2nd Law is:

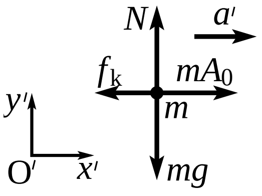
$$\vec{F}' = \vec{F} - m\vec{A}_0 = m\vec{a}\,',$$

where $\vec{F}$ are the “real forces” acting on m, $\vec{A}_0$ is the acceleration of O′ relative to any inertial frame, and $\vec{a}\,'$ is the acceleration of the box in frame O′, the quantity we seek. Thus, from FBD 1, we have:

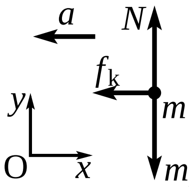
$$x'/ \quad -f_\text{k} + mA_0 = ma'; \tag{1}$$

$$y'/ \quad N - mg = 0; \quad N = mg. \tag{2}$$





FBD 1



FBD 2

But $f_\mathrm{k} = \mu_\mathrm{k} N = \mu_\mathrm{k} mg$, using (2). Thus, (1) becomes:

$$a' = A_0 - \mu_\mathrm{k} g = \frac{g}{2} - \frac{g}{3} = \frac{g}{6}.$$

Since $a' > 0$, the acceleration is *forward* (in direction of $+\hat{\imath}'$), as indicated in FBD 1.

b) In the frame of reference of the road (O), we have from FBD 2:

$$x/ \quad -f_\mathrm{k} = -ma; \qquad y/ \quad N - mg = 0; \quad N = mg.$$

Again, $f_\mathrm{k} = \mu_\mathrm{k} mg$ and the x component becomes:

$$a = \mu_\mathrm{k} g = \frac{g}{3}, \tag{3}$$

in the direction of $-\hat{\imath}$, as indicated in FBD 2. As a check, we have from 5.1.3 of the text (eds. 6 and 7),

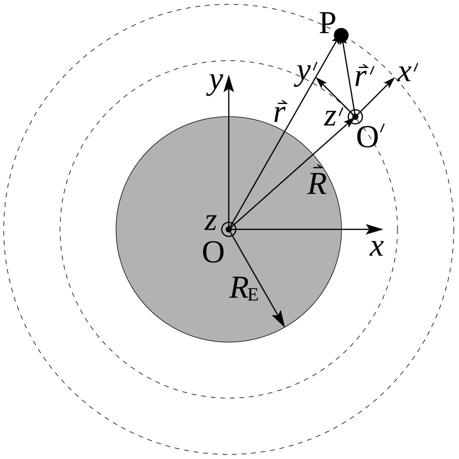
$$\vec{a} = \vec{A}_0 + \vec{a}' = \frac{g}{2}(-\hat{\imath}) + \frac{g}{6}\hat{\imath}' = -\frac{g}{3}\hat{\imath},$$

in agreement with (3), and where we used the fact that $\hat{\imath}' = \hat{\imath}$.

Tutorial 8.2

Problem 3 (FC 5.18) In the figure, objects P and O$'$ are "close-by" satellites in counter-clockwise circular orbits about the earth, where the z-axis is aligned with the north pole. Show that the acceleration of P relative to O$'$ is given by:

$$\ddot{x}' = 2\omega\dot{y}' + 3\omega^2 x'; \qquad \ddot{y}' = -2\omega\dot{x}'.$$



In the diagram, O is the origin of the *inertial* reference frame at the centre of the earth where the x-y axes do not rotate with the earth. The x'-y' axes are affixed to O$'$ and rotate about the earth with it.

Hint: To find the acceleration of P relative to O$'$, start with equation 5.2.14 (eds. 6 and 7), and assess each term as we've done in class. The only term that's zero is the $\dot{\omega}$ term. After doing some cross products, you should find:

$$\vec{a}' = (\omega^2 - \omega_\mathrm{P}^2)\vec{r} + 2\omega(\dot{y}'\hat{\imath}' - \dot{x}'\hat{\jmath}'),$$

where the first term on the RHS is proportional to $\vec{r}$ and not $\vec{r}\,'$. You'll also want Kepler's third law ($\omega^2 \propto r^{-3}$), and the fact that $r' \ll R$ will allow you to drop terms of order r'^2/R^2.

Solution: Let O be the inertial frame of reference whose origin is at the centre of the earth, and let ω and $\omega_{\rm P}$ be the angular speeds of O$'$ and P respectively about the earth. Next, write down the acceleration equation 5.2.14 (eds. 6 and 7) with $\dot\omega = 0$:

$$\vec a' = \vec a - \vec A - \vec\omega \times (\vec\omega \times \vec r') - 2\,\vec\omega \times \vec v', \tag{1}$$

which gives the acceleration of P relative to O$'$, the desired quantity.

Step 1. "Hunting and gathering"

$$\begin{aligned}
&\text{acceleration of P relative to O:} && \vec a = -\omega_{\rm P}^2 \vec r; && (2)\\
&\text{acceleration of O}' \text{ relative to O:} && \vec A = -\omega^2 \vec R; && (3)\\
&\text{angular velocity of O}' \text{ relative to O:} && \vec\omega = \omega\hat k';\\
&\text{position of P relative to O}'\text{:} && \vec r' = x'\hat\imath' + y'\hat\jmath';\\
&\text{velocity of P relative to O}'\text{:} && \vec v' = \dot x'\hat\imath' + \dot y'\hat\jmath'.
\end{aligned}$$

Step 2: Do the cross products.

$$\begin{aligned}
\text{centrifugal:}\quad & \vec\omega \times \vec r' = \omega\hat k' \times (x'\hat\imath' + y'\hat\jmath') = -\omega y'\hat\imath' + \omega x'\hat\jmath'\\
\Rightarrow\quad & \vec\omega \times (\vec\omega \times \vec r') = \omega\hat k' \times (-\omega y'\hat\imath' + \omega x'\hat\jmath') = -\omega^2 x'\hat\imath' - \omega^2 y'\hat\jmath' = -\omega^2 \vec r'; && (4)\\
\text{Coriolis:}\quad & \vec\omega \times \vec v' = \omega\hat k' \times (\dot x'\hat\imath' + \dot y'\hat\jmath') = -\omega\dot y'\hat\imath' + \omega\dot x'\hat\jmath'. && (5)
\end{aligned}$$

Step 3. Assemble the accelerations and do the analysis.

Substituting equations (2)–(5) into (1), we get:

$$\vec a' = -\omega_{\rm P}^2 \vec r + \underbrace{\omega^2 \vec R + \omega^2 \vec r'}_{\omega^2 \vec r} + 2\omega(\dot y'\hat\imath' - \dot x'\hat\jmath')$$

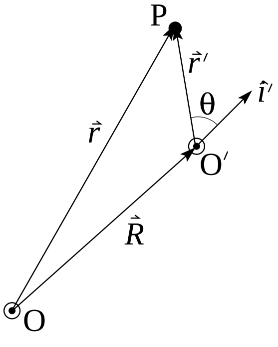
$$\Rightarrow \quad \vec a' = (\omega^2 - \omega_{\rm P}^2)\vec r + 2\omega(\dot y'\hat\imath' - \dot x'\hat\jmath'). \tag{6}$$

Then, according to Kepler's third law, we can write: $\dfrac{\omega_{\rm P}^2}{\omega^2} = \dfrac{R^3}{r^3}$. Thus,

$$\omega^2 - \omega_{\rm P}^2 = \omega^2 \left(1 - \frac{\omega_{\rm P}^2}{\omega^2}\right) = \omega^2 \left(1 - \frac{R^3}{r^3}\right). \tag{7}$$

Further, referring to the diagram to the right,

$$\begin{aligned}
r^2 &= \vec r \cdot \vec r = (\vec R + \vec r') \cdot (\vec R + \vec r')\\
&= R^2 + r'^2 + 2\vec R \cdot \vec r' = R^2 + r'^2 + 2R\underbrace{r'\cos\theta}_{x'}
\end{aligned}$$



$$= R^2\left(1+\frac{2x'}{R}+\frac{r'^2}{R^2}\right).$$

To now, everything has been exact. Let us now use the notion that O$'$ and P are "close-by" and set $r' \ll R$, and thus drop all terms of order r'^2/R^2. Thus,

$$r^2 \simeq R^2\left(1+\frac{2x'}{R}\right) \quad\Rightarrow\quad \frac{R^2}{r^2} \simeq \left(1+\frac{2x'}{R}\right)^{-1}$$
$$\Rightarrow \quad \frac{R^3}{r^3} \simeq \left(1+\frac{2x'}{R}\right)^{-3/2} \simeq 1-\frac{3x'}{R}, \tag{8}$$

where we performed a binomial expansion to two terms. Substitute equation (8) into equation (7) to get:

$$\omega^2-\omega_{\mathrm{P}}^2 \simeq \omega^2\left(1-1+\frac{3x'}{R}\right) = 3\omega^2\frac{x'}{R}. \tag{9}$$

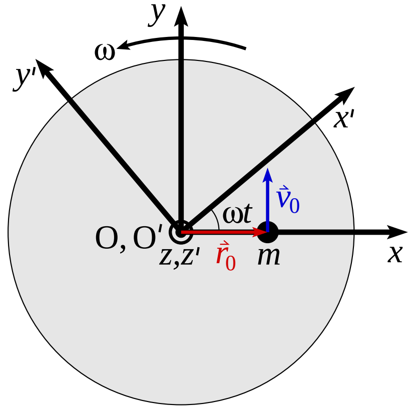
We then substitute equation (10) into equation (6) to get:

$$\vec a' \simeq 3\omega^2 x' \underbrace{\frac{\vec r}{R}}_{\sim\,\hat\imath'} + 2\omega(\dot y'\hat\imath' - \dot x'\hat\jmath') = (3\omega^2 x' + 2\omega\dot y')\hat\imath' - 2\omega\dot x'\hat\jmath'.$$

Breaking this up explicitly into its components then gives us the desired result.

Tutorial 8.3

Problem 4: A merry-go-round with a frictionless ice surface rotates with a constant angular velocity, ω. Relative to an inertial (non-rotating) reference frame, O, whose origin is at the rotation axis, a hockey puck is placed at position $r_0\hat\imath$ and given an initial velocity $v_0\hat\jmath$.



a) In a reference frame O$'$ rotating with the merry-go-round and whose origin coincides with O, show that the equations of motion for the puck are:

$$\ddot x' = \omega^2 x' + 2\omega\dot y'; \qquad \ddot y' = \omega^2 y' - 2\omega\dot x', \tag{1}$$

two coupled, second order differential equations. Yikes!

b) Write down the equations of motion of the puck relative to O, and solve these to find $x(t)$ and $y(t)$.

c) By performing the appropriate coordinate transformation between O and O$'$, find $x'(t)$ and $y'(t)$, and show that these solve differential equations (1).

Solution: a) Consider first the motion of the puck relative to O$'$.

Step 1: "Hunting and gathering".

position of m relative to O$'$: $\vec r\,' = x'\hat\imath\,' + y'\hat\jmath\,'$

velocity of m relative to O$'$: $\vec v\,' = \dot x'\hat\imath\,' + \dot y'\hat\jmath\,'$

acceleration of m relative to O$'$: $\vec a\,' = \ddot x'\hat\imath\,' + \ddot y'\hat\jmath\,'$

angular velocity of O$'$: $\vec\omega = \omega\hat k'$;

angular acceleration of O$'$: $\dot{\vec\omega} = 0$;

acceleration of O$'$ relative to O: $\vec A = 0$.

Step 2: Do the cross products.

$$\begin{aligned}
a_\perp &= -\dot{\vec\omega}\times\vec r\,' = 0;\\
a_{\text{Cor}} &= -2\vec\omega\times\vec v\,' = -2\omega\big[\dot x'(\hat k'\times\hat\imath\,') + \dot y'(\hat k'\times\hat\jmath\,')\big] = 2\omega(\dot y'\hat\imath\,' - \dot x'\hat\jmath\,');\\
a_{\text{cent}} &= -\vec\omega\times(\vec\omega\times\vec r\,') = -\omega^2\big[x'\big(\hat k'\times(\hat k'\times\hat\imath\,')\big) + y'\big(\hat k'\times(\hat k'\times\hat\jmath\,')\big)\big]\\
&= \omega^2(x'\hat\imath\,' + y'\hat\jmath\,') = \omega^2\vec r\,'.
\end{aligned}$$

Step 3: Assemble the accelerations.

Since there are no real forces acting on m in the horizontal plane, $\vec a = 0$ and,

$$\begin{aligned}
\vec a\,' &= \cancelto{0}{\vec a} + \cancelto{0}{\vec a_\perp} + \vec a_{\text{Cor}} + \vec a_{\text{cent}} + \cancelto{0}{\vec A}\\
&= 2\omega(\dot y'\hat\imath\,' - \dot x'\hat\jmath\,') + \omega^2(x'\hat\imath\,' + y'\hat\jmath\,') = (\omega^2x' + 2\omega\dot y')\hat\imath\,' + (\omega^2y' - 2\omega\dot x')\hat\jmath\,'\\
&\Rightarrow \boxed{\ddot x' = \omega^2x' + 2\omega\dot y' \qquad \text{and} \qquad \ddot y' = \omega^2y' - 2\omega\dot x',}
\end{aligned}$$

which are equations (1).

b) With no real forces acting on m, $\vec F = m\vec a = 0$. Thus, relative to inertial frame O,

$$\boxed{\ddot x = \ddot y = 0} \quad\Rightarrow\quad \boxed{\begin{aligned} x(t) &= x_0 + v_{0,x}t = r_0;\\ y(t) &= y_0 + v_{0,y}t = v_0t,\end{aligned}} \tag{2}$$

using the given initial conditions $\vec r_0 = (x_0, y_0) = (r_0, 0)$ and $\vec v_0 = (v_{0,x}, v_{0,x}) = (0, v_0)$.

c) The coordinate transformation between O and O′ is:

$$\begin{aligned} \hat{\imath} &= \hat{\imath}' \cos\omega t - \hat{\jmath}' \sin\omega t; \\ \hat{\jmath} &= \hat{\imath}' \sin\omega t + \hat{\jmath}' \cos\omega t. \end{aligned} \tag{3}$$

Thus, using equations (2) and (3), we can write down the position of m relative to O′:

$$\begin{aligned} \vec{r}\,' = \vec{r} = r_0\hat{\imath} + v_0 t\hat{\jmath} &= r_0(\hat{\imath}' \cos\omega t - \hat{\jmath}' \sin\omega t) + v_0 t(\hat{\imath}' \sin\omega t + \hat{\jmath}' \cos\omega t) \\ &= (r_0 \cos\omega t + v_0 t \sin\omega t)\hat{\imath}' + (-r_0 \sin\omega t + v_0 t \cos\omega t)\hat{\jmath}' \end{aligned}$$

$$\Rightarrow \quad \boxed{x'(t) = r_0 \cos\omega t + v_0 t \sin\omega t; \qquad y'(t) = -r_0 \sin\omega t + v_0 t \cos\omega t.} \tag{4}$$

Taking derivatives with respect to time, we get:

$$\dot{x}'(t) = -r_0\omega \sin\omega t + v_0 \sin\omega t + v_0\omega t \cos\omega t; \tag{5}$$

$$\ddot{x}'(t) = -r_0\omega^2 \cos\omega t + 2v_0\omega \cos\omega t - v_0\omega^2 t \sin\omega t; \tag{6}$$

$$\dot{y}'(t) = -r_0\omega \cos\omega t + v_0 \cos\omega t - v_0\omega t \sin\omega t; \tag{7}$$

$$\ddot{y}'(t) = r_0\omega^2 \sin\omega t - 2v_0\omega \sin\omega t - v_0\omega^2 t \cos\omega t. \tag{8}$$

Then, substituting the first of equations (4) and equation (7) into the RHS of the first of equations (1), we get:

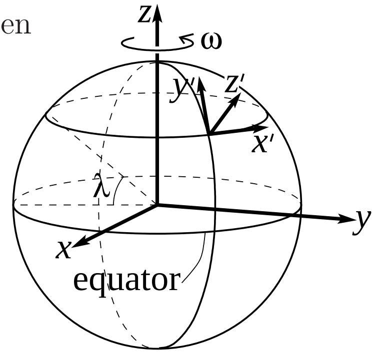
$$\begin{aligned} &\omega^2(r_0 \cos\omega t + v_0 t \sin\omega t) + 2\omega(-r_0\omega \cos\omega t + v_0 \cos\omega t - v_0\omega t \sin\omega t) \\ &= \cancel{r_0\omega^2 \cos\omega t} + \cancel{v_0\omega^2 t \sin\omega t} - \cancel{2}r_0\omega^2 \cos\omega t + 2v_0\omega \cos\omega t - \cancel{2}v_0\omega^2 t \sin\omega t \\ &= -r_0\omega^2 \cos\omega t + 2v_0\omega \cos\omega t - v_0\omega^2 t \sin\omega t = \ddot{x}'(t) \text{ (equation 6)} \\ &= \text{LHS of first of equations 1. } \checkmark \end{aligned}$$

Similarly, substituting the second of equations (4) and equation (5) into the RHS of the second of equations (1), we get:

$$\begin{aligned} &\omega^2(-r_0 \sin\omega t + v_0 t \cos\omega t) - 2\omega(-r_0\omega \sin\omega t + v_0 \sin\omega t + v_0\omega t \cos\omega t) \\ &= \cancel{-r_0\omega^2 \sin\omega t} + \cancel{v_0\omega^2 t \cos\omega t} + \cancel{2}r_0\omega^2 \sin\omega t - 2v_0\omega \sin\omega t - \cancel{2}v_0\omega^2 t \cos\omega t \\ &= r_0\omega^2 \sin\omega t - 2v_0\omega \sin\omega t - v_0\omega^2 t \cos\omega t = \ddot{y}'(t) \text{ (equation 8)} \\ &= \text{LHS of second of equations 1. } \checkmark \end{aligned}$$

Tutorial 8.4

Problem 5 (FC 5.12) A particle moves in a horizontal plane near the surface of the Earth. Show that the magnitude of the horizontal component of the Coriolis force is independent of the direction of motion.



Solution: If motion is restricted to the surface of the Earth, then

$$\vec{v}' = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'.$$

Meanwhile, from equation (5.4.9) of the text (eds. 6 and 7):

$$\vec{\omega} = \omega\hat{k} = \omega\cos\lambda\,\hat{\jmath}' + \omega\sin\lambda\,\hat{k}'.$$

Thus, the Coriolis force is given by:

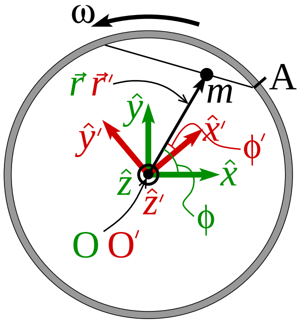
$$\begin{aligned}
-2m\vec{\omega}\times\vec{v}' &= -2m\omega\big(\cos\lambda\,\hat{\jmath}' + \sin\lambda\,\hat{k}'\big)\times\big(\dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'\big)\\
&= -2m\omega\big(\dot{x}'\cos\lambda\,\hat{\jmath}'\times\hat{\imath}' + \dot{x}'\sin\lambda\,\hat{k}'\times\hat{\imath}' + \dot{y}'\sin\lambda\,\hat{k}\times\hat{\jmath}'\big)\\
&= 2m\omega\dot{x}'\cos\lambda\hat{k} \qquad\qquad\qquad\quad \text{(vertical component)}\\
&\quad\; + 2m\omega\big(\dot{y}'\sin\lambda\,\hat{\imath}' - \dot{x}'\sin\lambda\,\hat{\jmath}'\big) \;\; \text{(horizontal component)}
\end{aligned}$$

The magnitude of the horizontal component is:

$$\big|(-2m\vec{\omega}\times\vec{v}')_{\text{hor}}\big| = 2m\omega\sin\lambda\sqrt{\dot{x}'^2 + \dot{y}'^2} = 2m\omega v'\sin\lambda,$$

which doesn't depend on the direction within the horizontal plane.

Problem 6 Consider again the O'Neill cylinder discussed in class and the same inhabitants launching projectiles at each other.



This time, consider the problem from the rotating reference frame O′ which, as shown in the figure, is cospatial with inertial reference frame O. If O′ rotates with the cylinder such that $\hat{x}'$ always points to A (fixed to the inside surface), show that the equations of motion for a projectile launched from A (with no external real forces acting on it) are given by:

$$\ddot{r}' = r'(\omega + \dot{\phi}')^2; \qquad \ddot{\phi}' = -\frac{2\dot{r}'}{r'}(\omega + \dot{\phi}'); \qquad \ddot{z}' = 0.$$

Solution: Start with the acceleration equation from the formula sheets:

$$\vec{a} \;=\; \vec{a}\,' + \dot{\vec{\omega}} \times \vec{r}\,' + 2\,\vec{\omega} \times \vec{v}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') + \vec{A}. \tag{9}$$

Step 1: "Hunting and gathering".

$$\begin{align}
\vec{a} &= 0 \quad \text{(no real forces acting on projectile)} \tag{10}\\
\vec{\omega} &= \omega\hat{z} \;=\; \omega\hat{z}' \nonumber\\
\dot{\vec{\omega}} &= 0 \tag{11}\\
\vec{r}\,' &= r'\hat{r}' + z'\hat{z}' \nonumber\\
\vec{v}\,' &= \dot{r}'\hat{r}' + r'\dot{\phi}'\hat{\phi}' + \dot{z}'\hat{z}' \nonumber\\
\vec{a}\,' &= (\ddot{r}' - r'\dot{\phi}'^2)\hat{r}' + (2\dot{r}'\dot{\phi}' + r'\ddot{\phi}')\hat{\phi}' + \ddot{z}'\hat{z}' \tag{12}\\
\vec{A} &= 0 \quad \text{(O and O$'$ are coincident)} \tag{13}
\end{align}$$

Step 2: Do the cross products.

$$\begin{align}
\vec{\omega} \times \vec{v}\,' &= \omega\hat{z}' \times (\dot{r}'\hat{r}' + r'\dot{\phi}'\hat{\phi}' + \dot{z}'\hat{z}') \;=\; \omega\dot{r}'\hat{\phi}' - \omega r'\dot{\phi}'\hat{r}', \tag{14}\\
\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') &= \omega^2\hat{z}' \times \big(\hat{z}' \times (r'\hat{r}' + z'\hat{z}')\big) \;=\; -\omega^2 r'\hat{r}'. \tag{15}
\end{align}$$

Step 3: Assemble the accelerations. Substitute equations (10)–(15) into equation (9) to get:

$$0 \;=\; (\ddot{r}' - r'\dot{\phi}'^2)\hat{r}' + (2\dot{r}'\dot{\phi}' + r'\ddot{\phi}')\hat{\phi}' + \ddot{z}'\hat{z}' + 2(\omega\dot{r}'\hat{\phi}' - \omega r'\dot{\phi}'\hat{r}') - \omega^2 r'\hat{r}'.$$

On equating components, we get:

$$\ddot{r}' \;=\; r'\dot{\phi}^2 + 2\omega r'\dot{\phi}' + r'\omega^2 \quad \Rightarrow \quad \boxed{\ddot{r}' \;=\; r'(\omega + \dot{\phi}')^2;}$$

$$r'\ddot{\phi}' \;=\; -2\dot{r}'\dot{\phi}' - 2\omega\dot{r}' \quad \Rightarrow \quad \boxed{\ddot{\phi}' \;=\; -\frac{2\dot{r}'}{r'}(\omega + \dot{\phi}');}$$

$$\text{and} \quad \boxed{\ddot{z}' \;=\; 0,}$$

as desired.
